# Towards a manifestly supersymmetric formulation of loop quantum supergravity theories

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Section 1

# Introduction: SUSY and LQG

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# The SUSY algebra

## Coleman-Mandula-Theorem (CM)

In the presence of a **mass gap**, the only possible (Lie) algebra of symmetries of the S-matrix for an interacting QFT is given by

$$\mathfrak{iso}(\mathbb{R}^{1,3}) \oplus \text{internal sym.}$$

## Haag-Łopuszański-Sohnius theorem

(CM) does not apply in case of **super Lie algebras**, i.e. $\mathbb{Z}_2$-graded algebras $(\mathfrak{g}, [\cdot,\cdot])$ of the form

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{with} \quad [\cdot,\cdot] : \text{(anti) commutator on } \mathfrak{g}_0 \ (\mathfrak{g}_1)$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (+ graded Jacobi identity)

# The SUSY algebra

$\Rightarrow$ smallest possible super Lie algebra of symmetries is given by the

## super Poincaré algebra ($\mathcal{N} = 1$ SUSY-algebra)

$$\mathfrak{iso}(\mathbb{R}^{1,3|4}) = \underbrace{\mathbb{R}^{1,3} \ltimes \mathfrak{so}(1,3)}_{\mathfrak{g}_0} \oplus \underbrace{S_{\mathbb{R}}}_{\mathfrak{g}_1}$$

generators: $P_I$, $M_{IJ}$, $Q^\alpha = (Q^A, Q_{A'})^T$ (Majorana spinor)

$$[P_I, \overline{Q}_\alpha] = 0 - \frac{1}{4L}\overline{Q}_\beta(\gamma_I)^\beta{}_\alpha$$

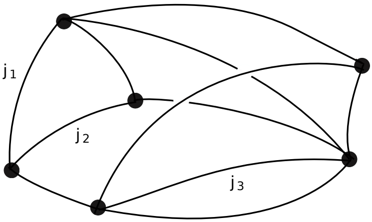
$$[M_{IJ}, \overline{Q}_\alpha] = \frac{1}{2}\overline{Q}_\beta(\gamma_{IJ})^\beta{}_\alpha$$

$$[\overline{Q}_\alpha, \overline{Q}_\beta] = \frac{1}{2}(C\gamma_I)_{\alpha\beta}P^I + \frac{1}{8L}(C\gamma^{IJ})_{\alpha\beta}M_{IJ}$$

$AdS_4 := \{x \in \mathbb{R}^5 | -(x^0)^2 + (x^1)^2 + \ldots + (x^3)^2 - (x^4)^2 = -L^2\}$

# LQG

- **nonperturbative** and **manifestly background independent** quantum theory of gravity
- starting point: reformulation of GR in terms of **Ashtekar-Barbero variables**
- gives canonical GR the structure of a **SU(2) Yang-Mills theory**
- Hilbert space: projective limit of $L^2$-functions on group-valued holonomies (parallel transport map) along one-dimensional paths
- Peter-Weyl theory $\Rightarrow$ basis: **spin networks** ($\rightarrow$ string-like excitations of the gravitational field)



# SUSY and LQG

**SUSY and LQG**:

- SUGRA in Ashtekar variables? [Füllöp, Jacobson, Pullin et al]
- Quantization [Smolin, Bodendorfer et al]

**But:**

- Canonical treatment of SUGRA theories generically suffer from complicated constraints
- SUSY hidden in the formalism $\rightarrow$ keep SUSY manifest (at least partially)?
- role of the Barbero-Immirzi parameter?

**What is needed for LQSG**:

- super Cartan geometry
- super analog of Ashtekar's connection
- anticommuting classical fermions
- super holonomies
- generalization of Peter-Weyl theory to super Lie groups

Section 2

# Supermanifolds and super Lie groups

# Supermanifolds

Various different approaches:

- **Berezin-Kostant-Leites**(algebro-geometric): "Definition like in NC geometry". But: what are the points?
- **Rogers-DeWitt**(concrete): "Start with topological space of points". But: Too many points, ambiguities"
- **Molotkov** '84 and **Sachse** '08(categorial): Shows BKL and RdW are two sides of the same coin.

**algebro-geom.**: Observation: Structure of a smooth manifold $M$ completely encoded in a suitable ring of functions on it
$\rightarrow$ describe $M$ as locally ringed space $(|M|, \mathcal{O}_M)$ s.t.

- $|M|$ paracompact topological Hausdorff space
- $\mathcal{O}_M$ abstract sheaf of local rings on $|M|$
- $M$ is locally Euclidean $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$

# Supermanifolds

## Definition Supermanifold

**Supermanifold** $\mathcal{M}$ is a locally ringed space $(M, \mathcal{O}_\mathcal{M})$ s.t.

- $M$ paracompact topological Hausdorff space
- $\mathcal{O}_\mathcal{M}$ abstract sheaf of local super rings on $M$
- $\mathcal{M}$ locally looks like flat superspace $\mathbb{R}^{m|n} = (\mathbb{R}^m, C^\infty_{\mathbb{R}^m} \otimes \bigwedge \mathbb{R}^n)$

- $\rightarrow$ locally, functions of the form:

$$f(x, \theta) = f_0(x) + f_i(x)\theta^i + \ldots + f_n(x)\theta^1 \cdots \theta^n$$

- $\mathcal{J} := \mathcal{O}_1 + \langle \mathcal{O}_1 \rangle^2$ (nilpotent sub ideal) $\rightarrow (M, \mathcal{O}_\mathcal{M}/\mathcal{J})$ ordinary manifold (body)
- **super Lie groups** as group objects in this category **SMan**.

# Supermanifolds

Supermanifold $\mathcal{M}$ yields *functor of points* $\mathcal{M} : \textbf{SMan}^{\mathrm{op}} \to \textbf{Set}$

$$\mathcal{T} \mapsto \mathcal{M}(\mathcal{T}) := \mathrm{Hom}_{\textbf{SMan}}(\mathcal{T}, \mathcal{M}) \quad (\mathcal{T}\text{-point})$$
$$(f : \mathcal{T} \to \mathcal{S}) \mapsto (\mathcal{M}(f) : g \mapsto g \circ f)$$

- restrict on *superpoints* $\mathcal{T} \cong (\{*\}, \Lambda)$ ($\to$ Grassmann algebras)

$$\mathcal{M}(\Lambda) \cong \mathrm{Hom}_{\textbf{SAlg}}(\mathcal{O}(\mathcal{M}), \Lambda)$$

- contains *real spectrum* $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) = \mathrm{Hom}_{\textbf{SAlg}}(\mathcal{O}(\mathcal{M}), \mathbb{R})$
- topological space via *Zariski* or *Gelfand topology*
- equip $\mathcal{M}(\Lambda)$ with coarsest topology s.t. $\mathcal{M}(\Lambda) \to \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ is continuous $\to$ **DeWitt-topology**

# Supermanifolds

- $\mathcal{M}(\Lambda_N)$ structure of topological manifold $\rightarrow$*Rogers-DeWitt supermanifold*
- $\Rightarrow$ yields functor

$$\mathbf{Gr} \rightarrow \mathbf{Man},\ \Lambda \mapsto \mathcal{M}(\Lambda)$$

- $\rightarrow$ *Supermanifold in the sense of Molotkov and Sachse*
- starting point for the construction of **infinite-dimensional supermanifolds** requiring $\mathcal{M}(\Lambda)$ to be a Banach [M '84, S '08] or Fréchet supermanifold [Schütt '19]
- $\rightarrow$ groups of super diffeomorphisms and supersymmetry transformations

Section 3

# Gravity as Cartan geometry

# Klein geometry

F. Klein: "Classify geometry of space via group symmetries".

Example: Minkowski spacetime $\mathbb{M} = (\mathbb{R}^{1,3}, \eta)$

- isometry group $\mathrm{ISO}(\mathbb{R}^{1,3}) = \mathbb{R}^{1,3} \ltimes \mathrm{SO}_0(1,3)$
- event $p \in \mathbb{M}$: $G_p = \mathrm{SO}_0(1,3)$ (isotropy subgroup)

$$\mathrm{ISO}(\mathbb{R}^{1,3})/\mathrm{SO}_0(1,3) \cong \mathbb{M}$$

**Definition**

A *Klein geometry* is a pair $(G, H)$ where $G$ is a Lie group and $H \subseteq G$ a closed subgroup such that $G/H$ is connected.

# Cartan geometry

- flat spacetime $\leftrightarrow$ metric Klein geometry
- $\Rightarrow$ Cartan geometry as deformed Klein geometry

## metric Cartan geometry

A **metric Cartan geometry** modeled on a metric Klein geometry $(G, H)$ is a principal $H$-bundle

$$\begin{array}{ccc} P & \xleftarrow{\;r\;} & H \\ \downarrow{\scriptstyle \pi} & & \\ M & & \end{array}$$

together with a **Cartan connection** $A \in \Omega^1(P, \mathfrak{g})$ s.t.

1. $r_g^* A = \mathrm{Ad}(g^{-1})A \ \forall g \in H$
2. $A(\widetilde{X}) = X, \ \forall X \in \mathfrak{h} \ \left(\widetilde{X} := (\mathbb{1} \otimes X_e) \circ r^*\right)$
3. $A: \ T_pP \to \mathfrak{g}$ is an isomorphism $\forall p \in P$

without (III): **generalized** Cartan geometry

# Cartan geometry

- Gravity as metric Cartan geometry $(M \overset{\pi}{\leftarrow} P, A)$ modeled on $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}_0(1,3))$

## Decomposition

$$A = \mathrm{pr}_{\mathbb{R}^{1,3}} \circ A + \mathrm{pr}_{\mathfrak{so}(1,3)} \circ A =: \xi + \omega$$

- $\omega$ Lorentz-connection
- $\xi \in \Omega^1_{\mathrm{hor}}(P, \mathbb{R}^{1,3})$ soldering form, induces isomorphism

$$\Xi : \; P \times_{(H,\mathrm{Ad})} \mathbb{R}^{1,3} \cong TM \Rightarrow \;\textbf{metric}\; g := \eta \circ (\Xi^{-1} \times \Xi^{-1}) \text{ on } M$$

## Action

$$S[A] = \int_M s^*(R[A]^{IJ} \wedge \xi^K \wedge \xi^L)\epsilon_{IJKL}$$

$s^*\xi =: e^I P_I \Rightarrow \{e^I\}$ coframe

Section 4

# Supergravity

# Supergravity

- Supergravity as **super Cartan geometry** modeled on super Klein geometry $(\mathrm{ISO}(\mathbb{R}^{1,3|4}), \mathrm{Spin}(1,3))$ [D'Auria-Fré-Regge '80, D'Auria-Fré '80, Castellani-D'Auria-Fré '91]

$$\begin{array}{ccc} \mathcal{P} & \longleftarrow & \mathrm{Spin}(1,3) \\ {\scriptstyle \pi}\downarrow & & \\ \mathcal{M} & & \end{array}$$

- super Cartan connection $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{iso}(\mathbb{R}^{1,3|4}))$
- Decompose

$$\mathcal{A} = \underbrace{\mathrm{pr}_{\mathfrak{g}_1} \circ \mathcal{A}}_{\psi} + \underbrace{\mathrm{pr}_{\mathbb{R}^{1,3}} \circ \mathcal{A}}_{\xi} + \underbrace{\mathrm{pr}_{\mathfrak{spin}(1,3)} \circ \mathcal{A}}_{\omega}$$

- $\psi = \psi^\alpha \overline{Q}_\alpha$ (spin-3/2) Rarita-Schwinger field

# Supergravity

**SUGRA action ($D = 4$, $\mathcal{N} = 1$)**

$$S[\mathcal{A}] = \int_M s^* \langle R[\mathcal{A}] \wedge \sigma \wedge \sigma \rangle \text{"="} S_P[e,\omega] + \int_M \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\sigma \gamma_* \mathcal{D}^\omega_\nu \psi_\rho$$

- $R[\mathcal{A}] := \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}]_{\mathfrak{g}}$
- $\sigma := \psi + \xi \in \Omega^1(\mathcal{P}, \mathfrak{t}^{1,3|4})$ super soldering form (super vielbein)
- $s : U \subset \mathcal{M} \to \mathcal{P}$ local section

Section 5

# Application for loop quantum supergravity

# Super Ashtekar connection

- self-dual Ashtekar connection:

$$A^{+IJ} = \frac{1}{2}\left(\omega^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{KL}\omega^{KL}\right) = (\Gamma^i - i\widetilde{K}^i)\, T_i^{+IJ}$$

- $T_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)$, $J_i = -\frac{1}{2}\epsilon_i{}^{jk} M_{jk}$ , $K_i = M_{0i}$

**Proposition: $(T_i^+, \overline{Q}_A)$ generate sub super Lie algebra**

$$[T_i^+, T_j^+] = \epsilon_{ij}{}^k T_k^+$$

$$[T_i^+, \overline{Q}_A] = -\frac{i}{2}\overline{Q}_B(\sigma_i)^B{}_A$$

$$[\overline{Q}_A, \overline{Q}_B] = 0 + \frac{1}{2L}(\epsilon\sigma^i)_{AB}\, T_i^+$$

generate **generalized Takiff Lie superalgebra** $\mathfrak{sl}(2,\mathbb{C}) \ltimes \mathbb{C}^2$
(orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)_{\mathbb{C}}$)

# Super Ashtekar connection

**Definition**

$$\mathcal{A}^+ := \psi^A \overline{Q}_A + A^{+i} T_i^+$$

$\Rightarrow$ Defines 1-form $\mathcal{A}^+ \in \Omega^1(\mathcal{P}, \mathfrak{osp}(1|2)_{\mathbb{C}})$

**Properties**

- $r_g^* \mathcal{A}^+ = \mathrm{Ad}(g^{-1})\mathcal{A}^+$, $\forall g \in \mathrm{SL}(2,\mathbb{C})$
- $\mathcal{A}^+(\widetilde{X}) = X$, $\forall X \in \mathfrak{sl}(2,\mathbb{C})$

Condition (III) not satisfied $\Rightarrow$ **generalized super Cartan connection**.

# Super Ashtekar connection

**Proposition:[KE '20]**

Consider associated $\mathrm{OSp}(1|2)_{\mathbb{C}}$-principal bundle

$$\begin{array}{ccc} \mathcal{P}\times_{\mathrm{SL}(2,\mathbb{C})}\mathrm{OSp}(1|2)_{\mathbb{C}} & \longleftarrow & \mathrm{OSp}(1|2)_{\mathbb{C}} \\ \downarrow{\scriptstyle\pi} & & \\ \mathcal{M} & & \end{array}$$

lift of $\mathcal{A}^+$ gives principal connection →**super Ashtekar connection**

- even exists for $\mathcal{N}>1$ (extended SUSY)
- what about **real** $\beta$?
  - $\Rightarrow$ both chiral components of $\overline{Q}$ in $\mathcal{A}^\beta$
  - But: $[\overline{Q}_A,\overline{Q}^{A'}]\propto P$ → no proper sub super algebra!
    → **no super holonomies**!

# Consequences

- $\mathcal{A}^+$ is the right starting point for LQSG
- fundamental description via super Cartan geometry
- $\rightarrow$ resolves confusions in literature:
  - existence with and without cosmological constant
  - requires chiral description of SUGRA (no real $\beta$)
- contains both gravity and matter d.o.f. $\rightarrow$ unified description, more fundamental way of quantizing fermions
- $\rightarrow$ can give **hint** how to quantize fermions in spin foam approach [Livine+Oeckl '03]
- substantially simplifies constraints (canonical form of Einstein equations): partial solution via gauge invariance[Fülöp '93, Ling+Smolin '00, Tsuda '00, Livine+Oeckl '03, KE+Sahlmann '20]

# Quantum theory

What do we need for quantum theory?

- holonomies (parallel transport map)
- Hilbert spaces
- spin network basis

**Holonomies**[KE '20 (in prep.)]:

- general problem: pullback of superfields to the body of a supermanifolds $\mathcal{M}$ are purely bosonic $\rightarrow$ **no fermionic degrees of freedom** on the underlying spacetime manifold!
- Resolution: Consider **enriched** category of supermanifolds [Schmitt '96, Deligne '99, Sachse '09, Groeger '14, Hack-Hanisch-Schenkel '15]
- Choose parametrizing supermanifold $\mathcal{S}$ and consider $\mathcal{S}$*-relative supermanifold* $\mathcal{M}/\mathcal{S} := (\mathcal{S} \times \mathcal{M}, \mathrm{pr}_{\mathcal{S}})$
- Have to require that physical quantities behave natural under change of parametrization $\lambda : \mathcal{S} \rightarrow \mathcal{S}'$

# Holonomies

- $\rightarrow$ Consider super connection 1-form $\mathcal{A}$ on $\mathcal{S}$-relative principal super fiber bundles
- yields parallel transport map along $\gamma : \mathcal{S} \times [0,1] \rightarrow \mathcal{M}$

$$\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma} : \Gamma(\gamma_0^* \mathcal{P}) \rightarrow \Gamma(\gamma_1^* \mathcal{P})$$

- natural under reparametrization: $\lambda^* \circ \mathscr{P}^{\mathcal{A}}_{\mathcal{S}',\gamma} = \mathscr{P}^{\lambda^*\mathcal{A}}_{\mathcal{S},\lambda^*\gamma} \circ \lambda^*$

**Super Wilson loop observable**

$$W_\gamma[\mathcal{A}] = \mathrm{str}\left( g_\gamma[\omega] \cdot \mathcal{P}\mathrm{exp}\left( -\oint_\gamma \mathrm{Ad}_{g_\gamma[\omega]^{-1}} \psi^{(\tilde{s})} \right)\right) : \mathcal{S} \rightarrow \mathcal{G}$$

- $\mathcal{A} = \omega + \psi$ and $\gamma : [0,1] \rightarrow M \subset \mathcal{M}$
- $g_\gamma[\omega]$: parallel transport map associated to $\omega$

# Holonomies

**Proposition**

- $W_\gamma[\mathcal{A}] : \mathcal{S} \to \mathcal{G}$ element in $\mathcal{G}(\mathcal{S})$ ($\mathcal{S}$-point of $\mathcal{G}$)
- natural under reparametrization $\lambda^* W_\gamma[\mathcal{A}] = W_\gamma[\lambda^* \mathcal{A}]$

- fermionic component $\psi \in \Omega^1(M, E) \otimes \mathcal{O}(\mathcal{S})_1$ describes **odd functional on supermanifold** $\mathcal{S}$
- choice of $\mathcal{S}$: $\mathcal{S}$ object in $\mathbf{SPt}^{\mathrm{op}} \cong \mathbf{Gr}$
  - $W_\gamma[\mathcal{A}] \in \mathcal{G}(\Lambda)$ (group element in Rogers-DeWitt supermanifold)
  - $\Lambda_\infty$ universal property: $\exists \lambda_N : \Lambda_N \hookrightarrow \Lambda_\infty \; \forall N \in \mathbb{N}_0$
- **Alternatively**: $\mathcal{S}$ as configuration space (infinite-dimensional smf!)

$$\mathcal{S} = \Omega^1(M, \mathrm{Ad}(P)) \oplus \Omega^1(M, E)$$

- $\Rightarrow \psi$ describes **odd functional on configuration space**[Schmitt '96, Rejzner '10]

# Hilbert space

**Super Peter Weyl**:

- use that super Lie groups have relatively simple structure, i.e. $\mathcal{O}_{\mathcal{G}} \cong C^\infty(G) \otimes \bigwedge \mathfrak{g}_1^* \Rightarrow$ invariant integrals of the form

$$\int_{\mathcal{G}} f = \int_G \mathrm{d}\mu_H \int_B \mathrm{d}\theta\, f(\theta)$$

- $\Rightarrow (\mathcal{O}_{\mathcal{G}}, \langle \cdot, \cdot \rangle)$ structure of **Krein space**
- for $\mathrm{U}(1|1)$ and $\mathrm{SU}(1|1)$ $(\mathfrak{u}(1|1)_0 = \mathfrak{u}(1) \oplus \mathfrak{u}(1))$ [KE '20 (in prep.)]

$$\overline{\mathcal{O}_{\mathcal{G}}}^{\|\cdot\|} \cong \bigoplus_{(m,n)\in\mathbb{Z}^2} \pi_{(m,n)}$$

Section 6

# Summary

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# Summary

- Considered supergravity as super Cartan geometry
- found that chiral generators $(T_i^+, \overline{Q}_A)$ generate **sub super Lie algebra** $\mathfrak{sl}(2,\mathbb{C}) \rtimes \mathbb{C}^2$ (or $\mathfrak{osp}(1|2)_{\mathbb{C}}$) of super Poincaré algebra
- gives rise to super Ashtekar connection $\mathcal{A}^+$ as **generalized super Cartan connection** (even for extended SUSY)
- $\mathcal{A}^+$ can be lifted to a super principal connection (à la Ehresmann) on associated principal super fiber bundle $\rightarrow$ **super holonomies**
- requires chiral description (no real $\beta$)
- considered parallel transport of super connection on parametrized supermanifolds

# Summary

- yields description of fermionic component of super connection as functional on supermanifold $\leftrightarrow$ description of fermions in pAQFT[Schmitt '96, Rejzner '10]
- $\Rightarrow$ in LQG: holonomies encode both matter and gravity d.o.f.
- Fermions: $1(+1)$ dimensional quantum excitations (as gravity).
- Construction of Hilbert space $\rightarrow$ derived super Peter Weyl theory for $\mathrm{SU}(1|1)$ and $\mathrm{U}(1|1)$.

**Outlook**

- non-manifestly supersymmetric approach to SUGRA:
  - Quantization of the SUSY-constraint $S$ (without $K$-term!) [KE+Sahlmann '20 (in prep.)]
  - since $\{S, S\} \propto H \Rightarrow$ consistency condition for Hamilton constraint
  - solutions of $S$ turn out to be really supersymmetric (**need** to contain both boson+fermion)
  - SUSY cosmology