A new construction of strict deformation quantization for Lagrangian fiber bundles

Mayuko Yamashita (RIMS, Kyoto)

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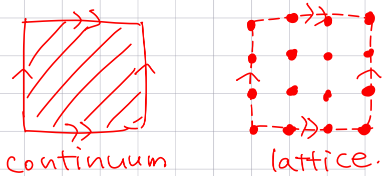
Today I will talk about my work to develop

$\boxed{\text{A Lattice version of Atiyah-Singer Index Theory}}$



- Motivated by Lattice Gauge theory
- A-S Index thm : D : elliptic diff. op
  → Ind(D) is computed from its principal symbol.

The story consists of :

1. Construction of a lattice version of the correspondence
   $\{\text{Diff. op}\} \longleftrightarrow \{\text{symbols}\}$
   - via Strict Deformation Quantization of torus bundles.
2. Formulation & proof of the Lattice Index theorem.

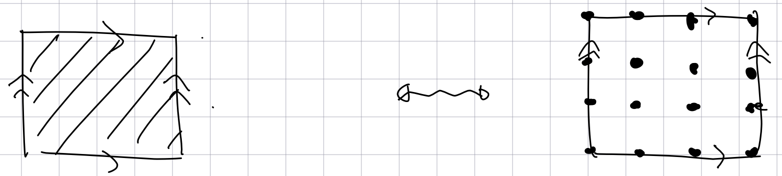
→ Application : The index problem of "Wilson-Dirac operator"
(from Lattice Gauge Theory)

§1. Overview

① Lattice Gauge theory :

Manifolds (spacetime) are approximated by lattice.

$$M \; (= \text{typically } \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n) \rightsquigarrow M_K \; (= \tfrac{1}{K}\mathbb{Z}^n/\mathbb{Z}^n).$$



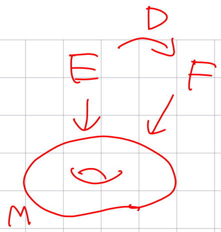
$$D^{\text{conti}} \curvearrowright L^2(M) \rightsquigarrow D^{\text{lat}}_K \curvearrowright \ell^2(M_K)$$

Need to recover information of D^{conti} (diff. op. on M) from its "lattice version" D^{lat}_K (op. on $\ell^2(M_K)$)

Today, we are interested in Index-type invariants :

(describing "anomaly" in physics)

Index theory on elliptic operators



- Continuum side :

$$D^{\mathrm{conti}} : L^2(M;E) \to L^2(M;F) \quad \text{elliptic diff op.}$$

$(M : \mathrm{cpt})$

$\to$ $\mathrm{Ind}(D^{\mathrm{conti}}) := \dim \mathrm{Ker}\, D^{\mathrm{conti}} - \dim \mathrm{Coker}\, D^{\mathrm{conti}}$

$\boxed{\text{Thm}}$ (Atiyah- Singer index theorem)

$$\underbrace{\mathrm{Ind}(D^{\mathrm{conti}})}_{\text{analytic index}} = \underbrace{\pi_![\sigma(D^{\mathrm{conti}})]}_{\text{topological index (K-theoretic)}}$$

[Here
$$\begin{array}{ccc} K^0(T^*M) & \xrightarrow{\pi_!} & K^0(pt) = \mathbb{Z} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ [\sigma(D^{\mathrm{conti}})] & \xmapsto[\text{AS}]{} & \mathrm{Ind}(D^{\mathrm{conti}}) \end{array}$$
]

$\rightsquigarrow$ $\mathrm{Ind}(D^{\mathrm{conti}})$ is a topological invariant computed from its principal symbols. $(\in C^\infty(T^*M))$

◦ lattice side: Given $D^{\text{conti}} \curvearrowright L^2(M;E)$

Q1. ◦ Can we find a nice counterpart $D_k^{\text{lat}} \curvearrowright \ell^2(M_k;E)$?

◦ Can we find a nice invariants $\widetilde{\text{Ind}}(D_k^{\text{lat}})$ of D_k^{lat} ?
corresponding to $\text{Ind}(D^{\text{conti}})$? ("$\text{Ind}(D_k^{\text{lat}}) = 0$" doesn't work!)

this is already studied by physisists.

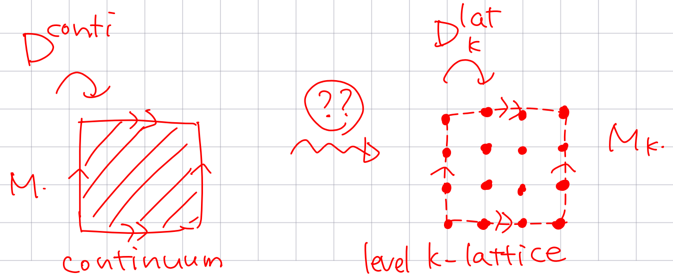
Q2. Do we have a topological formula for $\widetilde{\text{Ind}}(D_k^{\text{lat}})$?

(A lattice version of Atiyah-Singer index thm ?)

Q3 Do we have

$$\widetilde{\text{Ind}}(D_k^{\text{lat}}) \xrightarrow[k\to\infty]{} \text{Ind}(D^{\text{conti}}) \ ?$$

Today I will explain the solutions!



② Deformation Quantization.

$\boxed{\text{DQ Problem}}$

Given a symplectic manifold (X^{2n}, ω),

find a nice deformation $\{A_\hbar\}_{\hbar>0}$ of the Poisson alg $(C^\infty(X), \{,\})$

with $\phi_\hbar : C^\infty(X) \to A_\hbar$ linear maps

s.t. $[\phi_\hbar(f), \phi_\hbar(g)] = \hbar\, \phi_\hbar(\{f,g\}) + O(\hbar^2)$

※. Motivation :

Classical mechanics	Quantum mechanics
$\mathbb{R}^{2n}, (p,q)$	$L^2(\mathbb{R}^n)$
observables: $f \in C^\infty(\mathbb{R}^{2n})$	observables : operators on $L^2(\mathbb{R}^n)$
	CCR: $[q,p] = i\hbar$
p, q	$\longmapsto -i\hbar\frac{\partial}{\partial q}\ ,\ q.$
Generalize ⇓ (X^{2n}, ω) symplectic mfd.	?? $\rightsquigarrow$ Hilb. sp. $\mathcal{H}$.
$C^\infty(X) \ni f, g$	$[Op(f), Op(g)] \approx i\hbar\, Op(\{f,g\})$

The relation between Index theory and Deformation Quantization

… $\boxed{\text{Continuum side}}$ is well-known :

M : a cpt mfd $\rightsquigarrow$ (T^*M, ω_{can}) symplectic.

☆ $\Psi DO^*(M)_\hbar$ is a DQ for T^*M.

$$\begin{array}{ccc} C^\infty(T^*M) & \rightsquigarrow & \Psi DO^*(M)_\hbar \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma & \longmapsto & Op(\sigma) \end{array}$$

$\Downarrow$ Algebraic Index theorem (Nest-Tsygan '95)

☆ Atiyah-Singer index theorem

$$\begin{array}{ccc} K^0(T^*M) & \xrightarrow{\pi_!} & K^0(pt) = \mathbb{Z} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\sigma] & \longmapsto & \mathrm{Ind}(Op(\sigma)) \end{array}$$

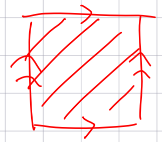
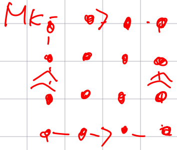
We are going to build the lattice ver. of this picture!

§2. Main Results | Lattice side |

$M = T^n$ (or more generally an integral affine manifold)

$\cup \quad\quad \cup$

$M_k := \frac{1}{k}\mathbb{Z}^n / \mathbb{Z}^n$ (level k- lattice)

M  M_k 

$T^*M/\Lambda^* \longrightarrow M$: $(\mathbb{R}/2\pi\mathbb{Z})^n$- bundle

(fiberwise lattice) (Lagrangian fiber bundle)

(*) A construction of strict DQ $\{\phi^k\}_k$. ($k = \frac{\sqrt{-1}}{\hbar}$)

$$\begin{array}{ccc} \phi^k : C^\infty(T^*M/\Lambda^*) & \longrightarrow & \mathbb{B}(\ell^2(M_k)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \phi^k(f). \end{array}$$

$\Downarrow$ Alg. Ind. Thm

(*) Thm (Lattice index theorem , Y)

$f \in M_N(C^\infty(T^*M/\Lambda^*))$, invertible self-adjoint, $k >> 0$

$$\Rightarrow \begin{array}{ccc} K^0(T^*M/\Lambda^*) & \xrightarrow{\pi_!} & K^0(pt) = \mathbb{Z} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [f] \otimes [L^k] & \longmapsto & \mathrm{rank}(E_{>0}(\phi^k(f))) \end{array}$$

Explanations :

⊛ Construction of strict DQ for $T^*M/\Lambda^* \xrightarrow{(\mathbb{R}/2\pi\mathbb{Z})^n} M$

(torus bundle)

… discrete version of the correspondence

$$\{\text{Diff - op}\} \longleftrightarrow \{\text{symbols}\}$$

Recall (continuum side)

$$\begin{array}{ccc} \mathrm{Diff}(M) & & C^\infty(T^*M) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ D & \rightsquigarrow & \sigma_{pr}(D) \end{array}$$

fiberwise Fourier transform

e.g. on $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

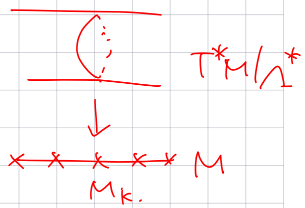
$$\left(q,\ i\frac{\partial}{\partial q}\right) \longleftrightarrow (q,\ p.)$$

$$\Downarrow$$

$\boxed{\text{Idea}}$ (lattice side) Define $\phi^k : C^\infty(T^*M/\Lambda^*) \to \mathbb{B}(\ell^2(M_k))$ by

$$\begin{array}{ccc} \mathbb{B}(\ell^2(M_k)) & & C^\infty(T^*M/\Lambda^*) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \phi^k(f) & \rightsquigarrow & f \end{array}$$

fiberwise Fourier expansion.



⊛ Lattice index theorem

Thm (Lattice index theorem, Y)

$f \in M_N(C^\infty(T^*M/\Lambda^*))$, invertible self-adjoint,

$\Rightarrow$ For $k >> 0$, $\phi^k(f) \in \mathbb{B}(\ell^2(M_k) \otimes \mathbb{C}^N)$ is invertible self-adjoint and

$$\begin{array}{ccc} K^0(T^*M/\Lambda^*) & \xrightarrow{\pi_!} & K^0(pt) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [f] \otimes [L^k] & \longmapsto & \operatorname{rank}(E_{>0}(\phi^k(f))) \end{array}$$

※ • $[L]$ is the class of pre-quantum line bundle on T^*M/Λ^*

… determined by the symplectic str on T^*M/Λ^*.

• $E \to M$ vector bundle

$\rightarrow$ We choose embedding $E \hookrightarrow M \times \mathbb{C}^N$ and apply the Thm

§3. Application : Index problem of Wilson-Dirac operators

Back to the motivation from Lattice Gauge Theory ...

Given $D^{conti} \curvearrowright L^2(M;E)$

Q1. ∘ Can we find a nice counterpart $D_k^{lat} \curvearrowright \ell^2(M_k;E)$?

∘ Can we find a nice invariants $\widetilde{\mathrm{Ind}}(D_k^{lat})$ of D_k^{lat} ?
corresponding to $\mathrm{Ind}(D^{conti})$? ($\mathrm{Ind}(D_k^{lat}) = 0$ doesn't work!)

this is already studied by physisists.

Q2. Do we have a topological formula for $\widetilde{\mathrm{Ind}}(D_k^{lat})$?

(A lattice version of Atiyah-Singer index thm ?)

Q3 Do we have

$$\widetilde{\mathrm{Ind}}(D_k^{lat}) \xrightarrow{k \to \infty} \mathrm{Ind}(D^{conti}) \ ?$$

Def For $\begin{cases} \cdot H: \text{finite dimensional Hilbert space} \\ \cdot D: \text{invertible self-adjoint operator on } H \end{cases}$

$\to$ $\widetilde{\mathrm{Ind}}(D) := \frac{1}{2}\left(\mathrm{rank}(E_{>0}(D)) - \mathrm{rank}(E_{<0}(D))\right)$

- Assume: $M = \mathbb{T}^{n}$ (even), $\$ \to M$: the spinor bundle.
  - $(F, \nabla) \to M$ a hermitian vec. bdl w/ a unitary conn.

<u>Notations</u>

- $\gamma \in \mathrm{End}(\$)$: the $\mathbb{Z}_2$-grading operator $(\gamma^2 = 1)$
- $c_i \in \mathrm{End}(\$)$: Clifford multiplications $(i = 1, \cdots, n)$
- Relation: $\gamma c_i + c_i \gamma = 0$, $\quad c_i c_j + c_j c_i = -2\delta_{ij}$

- We are interested in the twisted spin Dirac operator

$$D^{\text{conti}} : \Gamma(M; \$ \otimes F) \to \Gamma(M; \$ \otimes F)$$

$$D^{\text{conti}} := \sum_{i=1}^{n} c_i \nabla_i^{\$ \otimes F} \quad \leftarrow \text{ odd with respect to } \mathbb{Z}_2\text{-gr.}$$

$$D^{\text{conti}} = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix} \qquad \$ \underset{\gamma}{=} \$^+ \oplus \$^-$$

What is the nice lattice version $\{D^{\text{lat}}_k\}_k$?

- Physisists' suggestion : <u>Wilson Dirac operator</u>

Wilson Dirac operator (lattice side)

For $k \in \mathbb{Z}_{>0}$, $i \in \{1, \cdots, n\}$

let $\nabla_i : \ell^2(M_k; \$ \otimes F) \to \ell^2(M_k; \$ \otimes F)$ be the forward - differential

$$f \longmapsto (\nabla_i f)(x) = \frac{f(x + \frac{e_i}{k}) - f(x)}{\frac{1}{k}}$$

Def The Wilson-Dirac operator is defined by

$$\left| \begin{array}{l} D^{lat}_{W,k} : \ell^2(M_k; \$ \otimes F) \to \ell^2(M_k; \$ \otimes F) \\ D^{lat}_{W,k} := \sum_i \left\{ c_i \left(\frac{\nabla_i - \nabla_i^*}{2} \right) + \overbrace{\gamma \left(\frac{\nabla_i + \nabla_i^*}{2} \right)}^{* \text{ "Wilson term"}} \right\}. \end{array} \right.$$

$\Leftarrow$ self-adj, but NOT an odd operator : $\left(\begin{array}{c|c} * & D^+ \\ \hline D^- & * \end{array} \right)$

@ Physisist's Claim : For $m >> 0$,

$$\lim_{k \to \infty} \widetilde{\mathrm{Ind}} \left(D^{lat}_{W,k} + m\gamma \right) = \mathrm{Ind} \left(D^{conti} \right).$$

(Ginsberg-Wilson, ...)

<u>Our Result</u>. rigorous & topological proof for the Claim:

<u>Thm</u> (Adams, Kubota, Y., Fukaya-Furuta-Matsuo-Onogi-Yamaguchi-Y.)

For $m>0$ large enough,

$$\lim_{k\to\infty} \widetilde{\mathrm{Ind}}(D^{lat}_{w,k}+m\gamma) = \mathrm{Ind}(D^{conti})$$

i.e. lattice version of the index = continuum index !

The proof follows from a combination of Atiyah-Singer Index thm and Lattice Index thm.

<u>Key</u>: the commutativity of :

$$\begin{tikzcd} K^0(T^*M) \arrow[rr, "\pi_{T^*M}!"] \arrow[dr, "i_*"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & K^0(pt) = \mathbb{Z} \\ & K^0(T^*M/\Lambda^*) \arrow[ur, "\pi_{T^*M/\Lambda^*}!"'] & \end{tikzcd}$$

Sketch of the proof

- the discrete symbol $f_{DW} \in M_N(C^\infty(T^*M/\Lambda^*))$ is :

$$f_{DW} = \sum_i \left\{ c_i \left(\frac{e^{-\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_i}}{2} \right) + \gamma \left(\frac{e^{\sqrt{-1}\theta_i} - 2 + e^{-\sqrt{-1}\theta_i}}{2} \right) \right\},$$

$$\rightsquigarrow \quad D^{lat}_{W,k} = \frac{1}{k} \cdot \phi^k(f_{DW})$$

- We can show :

$$\begin{array}{ccc} K^0(T^*M) & \xrightarrow{i_!} & K^0(T^*M/\Lambda^*) \\ \Psi & & \Psi \\ [\sigma(D^{conti})] & \longmapsto & \left([f_{DW} + \frac{m}{k}\gamma] - [\gamma]\right) \otimes [L^k] \quad (\forall k) \end{array}$$

$\rightsquigarrow$

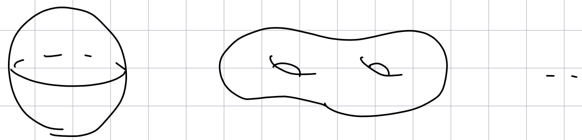
$$\begin{tikzcd} {[\sigma(D^{conti})]} \arrow[r, phantom, "\in"] \arrow[rrr, bend left, "\text{Atiyah-Singer}"] \arrow[dr, mapsto] & K^0(T^*M) \arrow[r, "\pi_{T^*M}!"] \arrow[d, "i_!"'] \arrow[dr, phantom, "\circlearrowleft"] & K^0(pt) & \mathrm{Ind}(D^{conti}) \overset{?}{=} \mathbb{Z} \\ & K^0(T^*M/\Lambda^*) \arrow[ur, "\pi_{T^*M/\Lambda^*}!"'] & & \widetilde{\mathrm{Ind}}(D^{lat}_{W,k} + m\gamma) \end{tikzcd}$$

$$\left([f_{DW} + \tfrac{1}{2}\gamma] - [\gamma]\right) \otimes [L^k] \in K^0(T^*M/\Lambda^*) \longmapsto \widetilde{\mathrm{Ind}}(D^{lat}_{W,k} + m\gamma)$$

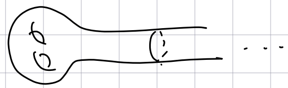
Lattice index thm

§4. Future directions

- Can we extend this theory to more general mfds?
  (i.e. mfds which does not admit integral affine str.)



- Non-compact case?
  - APS index ?



  - relation with coarse index theory ?