Spin$_c$ spectral geometry and fermion doubling

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The Standard Model

- One of the most successful and verified theories about the fundamental constituents of matter and forces between them.
- Nevertheless, some puzzles still remains unsolved: neutrinos, dark matter, ...
- Why the Higgs boson is *different*? Can all this be described geometrically?

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- {$C^* -$ algebra} $\equiv$ {ops. on a Hilbert space}
- differential structure (on closed, oriented Riemannian spin^c manifolds): the Dirac operator D acting on $L^2(M, S)$ determines the geodesic distance on M

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[Connes, 2013]

The metric and spin structure of a closed orientable Riemannian spin^c - manifold M can be encoded into a system

$$(C^\infty(M), L^2(M,S), D).$$

For $\dim M$ even, the are two important operators in the associated Clifford algebra Cl:

- *$\mathbb{Z}/2\mathbb{Z}$ - grading γ^5,*
- *charge conjugation operator C,*

that satisfy few compatibility conditions among themselves an the Dirac operator.

$$(A, H, D, \gamma, J)$$

- A - $*$ - algebra represented (in a faithful way) on a Hilbert space H
- $\mathbb{Z}/2\mathbb{Z}$ - grading γ on H s.th. $\gamma = \gamma^*$ and $[\gamma, A] = 0$
- antilinear isometry J
- selfadjoint operator D (with a compact resolvent)
- ... see e.g. [Lizzi, 2018]

Why spectral triples?

- Almost-commutative geometry ($C^\infty(M) \otimes A_f$) can explain the origin of the Higgs mass in the Standard Model. Furthermore, from the spectral action one can derive the form of an action for this model, and find relations between appropriate parameters. [Chamseddine-Connes-Marcoli, 2015]
- There are known successes in the derivation of Hilbert-Einstein action in Connes-Lott cosmology [Ackermann, 1996][Kastler,1995][Kalau-Walze, 1995]

- Start with a spectral triple (A, H, D, γ, J)
- Define $\Omega_D := \text{span}\{a[D, b] \ : a, b \in A\}$
- Consider $D_{\mathcal{A}} = D + \mathcal{A} + J\mathcal{A}J^{-1}$ for $\mathcal{A} \in \Omega_D$
- Define the bosonic spectral action

$$\mathcal{S}_B = \operatorname{Tr} f\left(\frac{D_{\mathcal{A}}}{\Lambda}\right),$$

  where f - smooth approximation of $\chi_{[0,1]}$ and Λ - cut-off parameter,
- ... and also the fermionic one: $\mathcal{S}_F = \langle\psi|D_{\mathcal{A}}|\psi\rangle$.
- Express $\mathcal{S}_B$ as an asymptotic series in Λ^{-1} and then the first term should give an effective action of the theory

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- Possibility of $SU(3)$-symmetry breaking
- Disagreement in the predicted Higgs mass and its experimental value

Possible partial solutions

- Lorenzian formulation [Paschke – Sitarz, 2006] [Barret, 2007] [Eckstein – Franco, 2014] [van den Dungen, 2015] [Brouder – Bizi – Besnard 2015] [Devastato – Farnsworth – Lizzi – Martinetti, 2018] [B. – Sitarz, 2018] [Martinetti – Singh, 2019] ...
- Fermion doubling problem [Lizzi – Mangano – Miele – Sparano, 1997] [Gracia-Bondia – Iochum – Schucker, 1998] [D'Andrea – Kurkov – Lizzi, 2016] ...
- Classification of finite Dirac operators and the problem of leptoquarks [Krajewski, 1998] [Paschke – Sitarz, 1998] [Paschke – Scheck – Sitarz, 1999] [Farnsworth – Boyle, 2014] [Dąbrowski – D'Andrea – Sitarz, 2018] [B. – Sitarz, 2018] ...
- New fermions approach [Stephan, 2006] [Stephan, 2007]
- σ field [Stephan, 2009] [Chamseddine – Connes, 2012] ...
- Grand Symmetry [Devastato – Lizzi – Martinetti, 2014] ...
- Twisted spectral triples [Landi – Martinetti, 2016] [Devastato – Martinetti, 2017] ...

All of the mentioned extensions are still based on the concept of (real) spectral triples with the product geometry. Furthermore, most of them similarly require cutting down the quadrupled number of the degrees of freedom.

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Our approach

We do not demand the almost-commutative structure, but rather a specific behaviour of the Lorentzian structure.

Krein-shifted geometry

- Dirac operator on the 4-dim Minkowski $(1,3)$ space: $D = i\gamma^\mu \partial_\mu$, where $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.
- Lorentz-invariant fermion action: $\int_M \bar{\psi} D \psi = \int_M \psi^\dagger \widetilde{D} \psi$, where $\bar{\psi} = \psi^\dagger \gamma^0$ and $\widetilde{D} = \gamma^0 D$ is the Krein shift of D.
- $\widetilde{D}$ is symmetric, while D is Krein self-adjoint: $D = \gamma^0 D \gamma^0$ with the fundamental symmetry γ^0. [Paschke – Sitarz, 2006] [B. – Sitarz, 2018]
- grading $\gamma = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$, real structure $\mathcal{J} = i\gamma^2 \circ cc$.
- $D\gamma = -\gamma D$, $D\mathcal{J} = \mathcal{J}D$, $\mathcal{J}^2 = 1$, $\mathcal{J}\gamma = -\gamma\mathcal{J}$.
- $\widetilde{D}\gamma = \gamma\widetilde{D}$, $\widetilde{D}\mathcal{J} = -\mathcal{J}\widetilde{D}$, $\mathcal{J}^2 = 1$, $\mathcal{J}\gamma = -\gamma\mathcal{J}$.

A Riemannian finite spectral triple built over a finite-dimensional algebra A is a collection of data (A, H, D, π_L, π_R), where π_L is a representation of A on H and π_R is a representation of A^{op} on H such that

- $[\pi_L(a), \pi_R(b)] = 0, \quad$ (0th order)
- $[[D, \pi_L(a)], \pi_R(b)] = 0 \quad$ (1st order)

for all a, b.

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We say that the triple if of

- spin$_c$ type if $(Cl_D(\pi_L(A)))' = \pi_R(A)$,
- Hodge type if $(Cl_D(\pi_L(A)))' = Cl_D(\pi_R(A))$,

where $Cl_D(\pi_L(A))$ is the algebra generated by $\pi_L(a)$ and $[D, \pi_L(a)]$ for all $a \in A$. Similarly we define $Cl_D(\pi_R(A))$.

This notion can be extended into genuine Riemannian geometries (requiring compact resolvent for D etc.) and also into the pseudo-Riemannian cases following the path of [B. – Sitarz, 2018].

Parametrization of the particle content in the one-generation Standard Model:

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W)$$

As an algebra we take $A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ with representations:

$$\pi_L(\lambda, q, m)\Psi = \begin{pmatrix} \lambda & & \\ & \bar{\lambda} & \\ & & q \end{pmatrix} \Psi, \qquad \pi_R(\lambda, q, m)\Psi = \Psi \begin{pmatrix} \lambda & \\ & m^T \end{pmatrix}.$$

At every point of the Minkowski space, we can encode any linear operator on the space of particles as an operator in $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$.

Lorentzian Dirac Operator

$$D_{\mathrm{SM}}\Psi = \underbrace{\begin{pmatrix} & & i\widetilde{\sigma}^\mu\partial_\mu & \\ & & & i\widetilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix}}_{D}\Psi + D_F\Psi,$$

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where D_F is a finite endomorphism of the Hilbert space $M_4(H_W)$.

- D is Lorentz-invariant, while D_F will transform covariantly only if is an element of $M_4(\mathbb{C}) \otimes \mathrm{id} \otimes M_4(\mathbb{C})$,
- If it is the case, D_F commutes with the chirality $\Gamma = \pi_L(1, -1, 1)$, so we have Lorentzian part D that anticommutes with Γ and a finite part that commutes with Γ. The Krein-shifted operators have opposite behaviour.

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- $\widetilde{D}$ alone obviously satisfies the order-one condition
- self-adjoint $\widetilde{D_F}$ are resctricted to

$$\widetilde{D_F} = \underbrace{\begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix}}_{D_l} \otimes e_{11} + \underbrace{\begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix}}_{D_q} \otimes (1_4 - e_{11}),$$

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Furthermore, analyzing the commutant of the generalized Clifford algebra generated by $\pi_L(a)$ and $[\widetilde{D_{\mathrm{SM}}}, \pi_L(a)]$ one can check that for one-generation model the spin$_c$ condition is satisfied.

The Hilbert space is now $M_4(H_W) \otimes \mathbb{C}^3$ with diagonal representation, so that M_l and M_q are now in $M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$.

- The spin$_c$ condition will hold if algebras generated by $\pi_L(A)$ and D_l, D_q, respectively, will be $(M_4(\mathbb{C}) \otimes \mathrm{id} \otimes \mathrm{id}) \otimes M_3(\mathbb{C})$ independently for the lepton and for quarks.
- Using similar argument as in [Dąbrowski – Sitarz, 2019] we infer the same condition for the Hodge property to be satisfied:

- Mass matrices:

$$M_l = \begin{pmatrix} \Upsilon_\nu & \\ & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & \\ & \Upsilon_d \end{pmatrix},$$

where Υ_e and Υ_u are chosen diagonal with the masses of electron, muon, and tau and the up, charm, and top quarks, respectively, and

$$\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger, \qquad \Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger$$

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- The sufficient condition to fulfill the Hodge property: for both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) their eigenvalues are pairwise different.
- This requirement is satisfied in the case of the physical standard model, provided that there is no massless neutrino [Dąbrowski – Sitarz, 2019]

- $(H_{\mathrm{SM}} = M_4(\mathbb{C}), \widetilde{D_F}, \Gamma)$ is an Euclidean even spectral triple.
- Suppose $\widetilde{D_F}$ is such that the spin$_c$ condition holds.
- Idea: *Doubling* of the triple $\Rightarrow$ The resulting real spectral triple will satisfy the Hodge duality and is the finite spectral triple of the SM studied so far as the finite component of the product geometry. :
- Take $H^2_{\mathrm{SM}} = H_{\mathrm{SM}} \oplus H_{\mathrm{SM}}$ with $\pi_L \oplus \pi_R$.
- Define $J(M_1 \oplus M_2) = M_2^* \oplus M_1^*$.
- Extend Γ so that $J\Gamma = \Gamma J$.
- $D' := \widetilde{D_F} \oplus 0 + J(\widetilde{D_F} \oplus 0)J^{-1}$.
- Then $D'\Gamma = -\Gamma D'$ and $D'J = JD'$.
- $Cl_{D'}((\pi_L \oplus \pi_R)(A)) = Cl_{\widetilde{D_F}}(\pi_L(A)) \oplus \pi_R(A)$.

Consider $M_4(H_W)$ with the real structure $\mathrm{id} \otimes \mathcal{J} \otimes \mathrm{id}$ (it means that J acts on finite part by complex conjugation). It does not implement the usual zero-order condition, but we still have a milder version:

$$\pi_R(A) \subseteq J\pi_L(A)J^{-1}.$$

- Imposing the same KO-dimension for the Euclidean finite spectral triple as for the Lorentzian spatial part, the mass matrices M_l and M_q have to be real.
- One generation: reality of fermion masses: OK
- Three generations: reality of PMNS and CKM matrices, CP symmetry preservation (vanishing of the Wolfenstein parameter $\bar{\eta}$ and CP-violating phase $\delta^{\nu}_{\mathrm{CP}}$; exp.: $\bar{\eta} = 0.355^{+0.012}_{-0.011}$ and $\delta^{\nu}_{\mathrm{CP}} = 1.38^{+0.52}_{-0.38}$ [PDG, 2018])
- The existence of CP-violation may be interpreted as a shadow of J-symmetry violation in the nondoubled spectral triple.

- The Krein-shifted Dirac operator $\widetilde{D_{SM}}$ satisfies 1st order condition
- The Lorentzian Dirac operator $D_{SM} = \beta\widetilde{D_{SM}}$ satisfied twisted condition:

$$[[D_{SM}, \pi_L(a)]_\beta, \pi_R(b)]_\beta = 0,$$

where $\beta = \mathrm{id} \otimes \gamma^0 \otimes \mathrm{id}$ and $[x, y]_\beta = xy - \beta y \beta^{-1} x$.

Summary

- We proposed a geometry of the SM which is not a product of spectral triples.
- When restricted to the commutative algebra of real-valued functions, we get even real Lorentzian spectral triple of KO-dimension 6 and with the Dirac operator satisfying the 1st order condition
- The restriction of the spectral triple to the constant functions over the Minkowski space gives a Euclidean even spectral triple, which fails to be real.
- The lack of real structure can be related to the appearance of the violation of CP symmetry in the SM
- The full spectral triple satisfies the spin$_c$ condition: the Clifford algebra generated by the commutators of the Krein-shifted Dirac operator with the representation π_L of the algebra has, as the commutant, the right representation of the algebra π_R.

Outlook

- What is the best description of this geometry? There is no product here but rather *a quotient spectral geometry*.
- Classify all such possible geometries as extensions of the SM.
- Compute the spectral action for such geometries [B.– Sitarz – Zalecki, in preparation] and compare values of parameters with experimental data.
- Beyond SM: application to e.g. Pati-Salam models with Lorentzian structures [B. – Williams – Zalecki, 2020] [B. – Williams – Zalecki, in preparation]
- ...

Thank you for your attention!