Heredegen's algebraic approach to Casimir effect

What two plates and two delta like systems tells as about?

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General remarks

The main aim of this talk is to present foundations of Herdegen's framework and interesting results obtained in it. On the mathematical side only main steps are presented, often in simplified form. Also some important results are only mention or even omitted.

All results presented here are derived in as rigorous way as of today's mathematics requires. Precise definitions, results and they proofs can be found in the works listed at the end of presentation.

I assume general knowledge of algebraic approach to quantum physics, but for the sake of clarity I will repeat well know points, stressing this that are important here.

Casimir effect in algebraic settings

Algebraic formalism

In this talk quantum system will be described by three elements.

1) C^* algebra $\mathcal{A}$.

2) Representation π of algebra $\mathcal{A}$ as operators acting on Hilbert space $\mathcal{H}$.

$$\pi : \mathcal{A} \to \pi(\mathcal{A}) \tag{1}$$

3) Time evolution implemented by family of automorphism of $\mathcal{A}$ and unitary transformations of $\mathcal{H}$.

$$\alpha_t : \mathcal{A} \to \mathcal{A}, \quad A \mapsto \alpha_t A, \quad A \in \mathcal{A}$$
$$\pi(\alpha_t A) = U(t)\pi(A)U^*(t), \quad U(t) = \exp(itH),$$

where H is hamiltonian of the system. We assume that it is nonnegative.

Algebraic formalism

Two systems Q_0 and Q_1 are physically comparable if and only if they have common algebra $\mathcal{A}$ and equivalent representations π_0 and π_1. If not, comparison of such systems can leads to "unreasonably" phenomena of physical origin, like physically justified infinities.

Consider for Minkowski spacetime at thermal equilibrium with $T = 0\,\mathrm{K}$ and $T = 1\,\mathrm{K}$. Transition from first state to the second require infinite amount of energy delivered to the system, so we shouldn't be surprised if compering this two states leads to infinite results.

We call interaction V **singular** if it introduction to the system change representation to nonequivalent to π_0. In other cases we call it **nonsingular**.

Algebraic problems of two "plates" system [Her01]

Consider massless scalar field ϕ in $1 + 1$ dimensions ($1 + 3$ is analogous) with two metal "plates" at distance $x = 0$ and $x = a$. In standard analysis of Casimir effect we require that field obey Dirchlet boundary conditions.

$$\partial_t^2 \phi(x, t) = -\Delta\phi(x, t) \tag{2}$$

$$\phi(0) = \phi(a) = 0 \tag{3}$$

How algebra C^* for such case looks like?

Algebraic problems of two plate system [Her01]

In current context algebra for two “plates” with Dirichlet boundary conditions should be build on the top of symplectic space of functions from $L^2(\mathbb{R}, dx)$, that are regular enough and vanish at the “plates” positions 0 and a (more precisely there are sum of functions from H_0^1). But this can give us different algebraic model for every value of a!

Situation is even worse when we want to find common algebra for plate position vary in range $(-l, l)$ and $(a - \varepsilon, a + \varepsilon)$. The functions must now vanish on both these intervals, which give us trivial theory.

Maybe I will have time go back to this problem.

Moral of the story

Dirichlet boundary conditions belongs to singular class of interactions. I will argue that for these reason we shouldn't be surprised be appearing of infinities in such *linear* system, since they are probably physical.

Also, renormalization of physical inifinites can lead to obscuring or even losing of important information's about system.

In the story part I will sketch Herdegen's formalism that allow us to investigate systems free from all this problems, which are "close enough" to Dirichlet case to recover canonical Casimir term. I will also explain what mean "close enough".

Herdegen’s approach

Herdegen framework. Assumptions [Her05]

Let Q be quantum system with Hamiltonian H_0 and M be macroscopic system described by classical variables collectively denoted a. For example a can by just separation of the plates. We make following assumptions about they interaction.

1) Introduction of system M is nonsingular for the Q.

2) When M is in fixed state described by a_1 time evolution is given by

$$\alpha_{a_1,t}\mathcal{A} \mapsto \mathcal{A}, \tag{4}$$

$$\pi(\alpha_{a_1,t}A) = U_{a_1}(t)\pi(A)U^*_{a_1}(t), \quad U_{a_1}(t) = \exp(itH_{a_1}). \tag{5}$$

Since we can always add $C\,\mathrm{id}$ to Hamiltonian there is big ambiguity in definition of H_a:

$$H_a \to H_a + C(a)\,\mathrm{id}, \tag{6}$$

with arbitrary function $C(a)$. Fortunately, this not introduce ambiguity to physical predictions.

Herdegen framework. Assumptions [Her05]

3) We will consider only **adiabatic** change of parameters $a(t)$.

4) When joint system Q-M evolves, energy of Q is as previously given by H_0, but due to interactions with M quantum states evolution is given by family of operators $H_M(t)$.

5) Due to adiabatic changes of $a(t)$ we can use approximation

$$H_M(t) = H_{a(t)}, \tag{7}$$

where $H_{a(t)}$ was defined previously as Hamiltonian for fixed state of M with parameters $a(t)$.

Herdegen framework. Assumptions [Her05]

6) For fixed t consider eigenstate of $H_{a(t)}$

$$H_{a(t)}\psi_{a(t)} = E_{a(t)}\psi_{a(t)}, \quad \text{for fixed } t. \tag{8}$$

We assume that this eigenstates are not degenerated and depend continuously on $a(t)$. Since we are interested in ground state this is not very restrictive assumption. Adiabatic evolution $\psi(t)$ of state $\psi_{a(0)}$ is given by

$$\psi(t) = \exp\left(i\varphi(E_{a(t)}, \psi_{a(t)}, t)\right)\psi_{a(t)}. \tag{9}$$

7) Expectation value of any observable O in the state $\psi_{a(t)}$ is given by

$$\langle O\rangle_t = (\psi_{a(t)}, O\psi_{a(t)}). \tag{10}$$

This expression is independent of any ambiguity in choosing of $H_{a(t)}$.

Herdegen framework. Assumptions [Her05]

8) Casimir energy is given by expectation value of H_0 (Hamiltonian of isolated system Q) in its current ground state $\Omega_{a(t)}$.

$$\mathcal{E}_{a(t)} = (\Omega_{a(t)}, H_0 \Omega_{a(t)}) \tag{11}$$

9) Casimir force is minus derivative of energy with respect to $a(t)$.

$$\mathcal{F}_{a(t)} = -\frac{\partial \mathcal{E}_{a(t)}}{\partial a(t)} \tag{12}$$

Since we are interested in how energy and force depend on a, not on t, from this moment I will omitting time dependence in a.

Constructions of algebra and representation [Her05]

This was list of our wishes, but if they can't come true, it is void. Fortunately there exist quit general construction.

We need *real* Hilbert space $\mathcal{R}$ and operator selfadjoint operator h with domain $D(h)$. Operator h need to be strictly positive or nonnegative (more about this latter): $h > 0$ or $h \geq 0$.

Consider $\mathcal{L} = D(h) \oplus \mathcal{R} \subset \mathcal{R} \oplus \mathcal{R}$. We denote $V_i = v_i \oplus u_i$, $v_i \in D(h)$, $u_i \in \mathcal{R}$. $\mathcal{L}$ is symplectic space with symplectic product:

$$\sigma(V_1, V_2) = (v_2, u_1) - (v_1, u_2). \tag{13}$$

Dynamics on $\mathcal{L}$ is given by Hamiltonian

$$\mathcal{H}(V) = \frac{1}{2}[(hv, hv) + (u, u)]. \tag{14}$$

We can solve equation of motions for this system and find time evolution family of transformations T_t.

Constructions of algebra and representation [Her05]

Let $\mathcal{K} = \mathcal{R} \oplus i\mathcal{R}$ be complexification of $\mathcal{R}$. We define operator

$$j : \mathcal{L} \to \mathcal{K},$$
$$j(V) = j(v \oplus u) = h^{1/2}v - ih^{-1/2}u.$$

Operator j requires that $h > 0$. In special cases we can allow it to have 0 eigenvalue.

After making this more precise and putting some work we can use standard construction of Weyl C^* algebra corresponding to symplectic space $\mathcal{L}$ with representation in Fock space with $\mathcal{K}$ as "one-particle space" (see the second volume of Bratteli, Robinson book [BR87]).

When you introduce macroscopic bodies M we only need to change operator h to h_a and repeat all above construction. This is easy to say, but hard to follow.

Energy and number of particles [Her05]

Ground state representations for algebraic systems constructed this way are equivalent if and only if

$$\mathcal{N}_a = (\Omega_a, N\Omega_a) = \mathrm{Tr}\left[h^{-1/2}(h_a - h)h_a^{-1}(h_a - h)h^{-1/2}\right] < \infty. \qquad (15)$$

In the context of quantum field this formula have clear meaning: introduction of macroscopic bodies create only finite number of free particles.

Physics requires that Casimir energy is also finite.

$$\mathcal{E}_a = (\Omega_a, H_0\Omega_a) = \mathrm{Tr}\left[(h_a - h)h_a^{-1}(h_a - h)\right] < \infty \qquad (16)$$

Proving of r.h.s. of this formulas takes some time.

Local energy density for quantum field [Her05]

In this case ground state Ω_a defines distribution on pairs of test functions f, g with L^2 scalar product.

$$T_a(f,g) = \frac{1}{4}\big(f, (h_a - h)g\big) + \frac{1}{4}\big(\nabla f, (h_a^{-1} - h^{-1})\nabla g\big) \tag{17}$$

By definition, outside singular support of distribution this give us kernel function $T_a(\vec{x}, \vec{y})$. In such region, and *only* in it, we can defined local energy density:

$$\varepsilon_a(\vec{x}) = T_a(\vec{x}, \vec{x}). \tag{18}$$

What this framework give us?

- We can recover in rigorous manner textbook results for two plates.
- Since energy is equal $(\Omega_a, H_0\Omega_a)$ it must be positive. There is no dubious “negative energy”.
- All physical quantities are under control.
- We can quite well understand relation between global and local energy.
- It is hard to find nonsingular interactions that looks simple and “natural”.
- To understand how to use this formalism you need to put some work.

What you need to do in practice?

1) Find promising system described by operators h and h_a. Show that $\mathcal{N}_a < \infty$ and $\mathcal{E}_a < \infty$.

2) Introduce rescaled family of systems $h_{\lambda,\,a}$, $\lambda \in [1, 0)$ such that $h_{1,\,a} = h_a$.

3) Show that for all $\lambda \in [1, 0)$ you have $\mathcal{N}_{\lambda,\,a} < \infty$, $\mathcal{E}_{\lambda,\,a} < \infty$.

4) Show that in limit $\lambda \searrow 0$ your family tends in the sens of resolvent limit (or other well defined way) to interesting system: $h_{\lambda,\,a} \to h_I$. This limit case is most probably singular.

5) Derive asymptotic expansion of Casimir energy $\mathcal{E}_{\lambda,\,a}$ around $\lambda = 0$.

6) Compute energy density $\varepsilon_{\lambda,\,a}(\vec{x})$. Distribution $T_{\lambda,\,a}(f, g)$ most probably will be regular on $\mathbb{R}^3$.

7) Compute energy density limit for $\lambda \searrow 0$. If limit case is singular, singular support of limit distribution will be probably nonempty.

Solved systems

- Multidimensional harmonic oscillator, Andrzej Herdegen, 2005, [Her05].
- Two plate system for scalar and electromagnetic field, Herdegen, 2006, [Her06], Andrzej Herdegen and Mariusz Stopa 2010, [HS10].
- Two point like objects for scalar field, Ziemian, 2020, to be published.

Two plates system

Herdegen and Stopa approach to two plates [HS10]

Plates are parallel to xy plane, so interesting dynamics is only in z directions. After "integreting out xy directions over unit area", consistency conditions take form

$$\mathrm{Tr}\left[(h_{z,a} - h_z)^2\right] < \infty, \tag{19}$$

$$\mathrm{Tr}\left[(h_{z,a} - h_z)h_z(h_{z,z} - h_z)\right] < \infty. \tag{20}$$

Energy per unit area

$$E_a = \frac{1}{24\pi}\mathrm{Tr}\left[(h_{z,a} - h_z)(2h_{z,a} + h_{z,a})(h_{z,a} - h_z)\right] \tag{21}$$

In this special case, since we don't have take h_a^{-1}, it can have 0 eigenvalue.

Herdegen and Stopa approach to two plates [HS10]

We will consider simpler case of scalar field. We have $\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3, d^3x)$, $\mathcal{K} = L^2(\mathbb{R}^3, d^3x)$, $h = \sqrt{-\Delta}$.

$$h_a^2 = -\Delta + V \tag{22}$$

V is projection operator of finite rank. In position space it is integral operator with kernel

$$V(z, y) = \left[f(z-b)\overline{f(y-b)} + f(z+b)\overline{f(y+b)} \right]. \tag{23}$$

$f(z)$ is smooth complex function with compact support.

Rescaled system is defined by

$$f_\lambda(z) = \lambda^{-1} f\left(\frac{z}{\lambda}\right) \tag{24}$$

For $\lambda \searrow 0$ system converge in resolvent sens to field with Dirichlet boundary conditions in planes $z = \pm b$.

Herdegen and Stopa approach to two plates [HS10]

Asymptotic expansion of Casimir energy

$$E_{\lambda,a} = \frac{E_\infty}{\lambda^3} + \frac{c}{\lambda a^2} - \frac{\pi^2}{1440 a^3} + O(\lambda), \tag{25}$$

where E_∞, c are constant.

E_∞/λ^3 is two times energy of single plate in vacuum, per unit area, $-\pi^2/1440a^3$ this term in this expansion recovers standard formula for Casimir force.

Constant c depends on choice of function f and typically $c > 0$, so force become repulsive for large a.

Since λ is roughly thickens of the plate, this model is only valid when $\lambda < a$ and this guaranties that energy is always positive.

Herdegen and Stopa approach to two plates [HS10]

For $\lambda \neq 0$ local energy density $\varepsilon_{\lambda, a}(\vec{x})$ is defined on whole $\mathbb{R}^3$. In the limit $\lambda \searrow 0$ singular support of distribution is equal to $\{-b, +b\}$ (on the x axis).

For this reason integration over whole space is mathematically unjustified. But, if we do it anyway we arrive at result

$$\int \varepsilon_{0, a}(x)\, dx = -\frac{\pi^2}{1440 a^3}. \tag{26}$$

This expression give us canonical Casimir term for force, but not recover total formula. It is also negative, while global energy is always positive.

Two delta like systems

Two delta like systems

In this case we know only results for scalar field.

This problem is in many ways like two plates problem.
$\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3, d^3x)$, $\mathcal{K} = L^2(\mathbb{R}^3, d^3x)$, $h = \sqrt{-\Delta}$.

$$h^2_{\vec{a}} = -\Delta + V \tag{27}$$

V is again projection operator of finite rank.

$$V(\vec{x}, \vec{y}) = \sigma(g) \left[g(\vec{x} - \vec{b})\overline{g(\vec{x} - \vec{b})} + g(\vec{x} + \vec{b})\overline{g(\vec{x} + \vec{b})} \right] \tag{28}$$

$g(\vec{x})$ is smooth, positive, spherical symmetric function with compact support.

Rescaled version of the model

$$g_\lambda(\vec{x}) = \lambda^{-3} g\left(\frac{\vec{x}}{\lambda}\right) \tag{29}$$

In limit $\lambda \searrow 0$ we have two δ system, well know in literature [Alb88].

Two delta like systems

Asymptotic expansion of Casimir energy

$$\mathcal{E}(a,\lambda) = \mathcal{E}_{\rm self}(\lambda) + \frac{2\alpha}{\pi^3}\left[\frac{\chi}{\lambda}\int\limits_0^{+\infty}\frac{e^{-2l}\,dl}{(\gamma+l)[(\gamma+l)^2-e^{-2l}]} + \right.$$

$$+\frac{b_1\chi}{\gamma}\int\limits_0^{+\infty}\frac{l^2[3(\gamma+l)^2e^{-2l}-e^{-4l}]}{(\gamma+l)^2[(\gamma+l)^2-e^{-2l}]^2}dl - \frac{2}{\gamma}\int\limits_0^{+\infty}\frac{le^{-2l}\,dl}{(\gamma+l)[(\gamma+l)^2-e^{-2l}]} +$$

$$\left. + \frac{1}{\gamma}\int\limits_0^{+\infty}\frac{(1-l)e^{-2l}}{(\gamma+l)^2-e^{-2l}}\,dl\right] + O(\lambda) \tag{30}$$

$\mathcal{E}_{\rm self}(\lambda)$ is two times energy of single δ in vacuum,
a is distance between centers of two bodies,
α is parameter of δ interaction [Alb88],
$\gamma = \alpha a/2\pi^2 > 1$ is constrain on distance a [Alb88],
$\chi > 0$ and $b_1 > 0$ are constants describing properties of function g.

Two delta like systems

Presented expression for global energy is dominated by model dependent terms which can't be removed. Numerical analysis show that force is repulsive. This contradict previous result found in literature that predicts universal attractive force [Sca05].

Also we can show that local energy density have universal limit, which exclude possibility of recovering global energy by integrating local [FP18], [Fer19].

Open problems

- How to compute electromagnetic Casimir effect for two delta like objects?
- How this framework relates to alternative algebraic formalism of Claudio Dappiaggi, Gabriele Nosari and Nicola Pinamonti [DNP16]?
- Investigate limit $\lambda \searrow 0$ for single sphere.

Thank you. Any question?

Papers on Herdegen's approach

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