About the correspondence between vertex operator superalgebras and graded-local conformal nets

Tiziano Gaudio
joint work with S. Carpi and R. Hillier (in preparation)

Lancaster University

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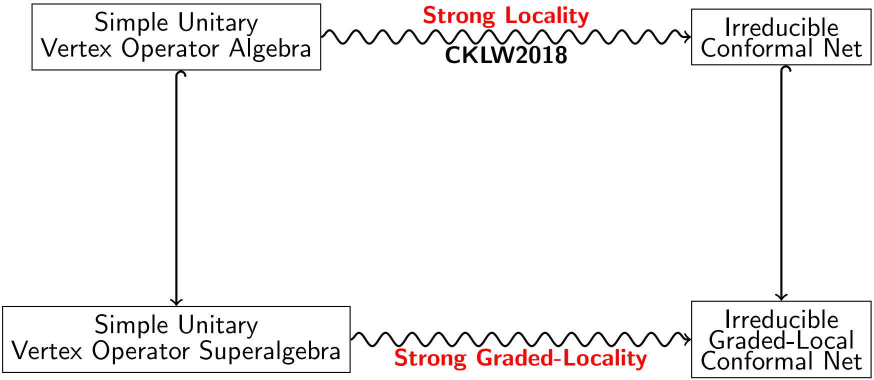
Introduction



Starting definitions

Let $\mathcal{J}$ be the set of all intervals (open, connected, non-empty and non-dense subsets) of the circle S^1. An **irreducible graded-local conformal net** is a family $\mathcal{A} := (\mathcal{A}(I))_{I\in\mathcal{J}}$ of von Neumann algebras on a separable Hilbert space $\mathcal{H}$, s.t.:

Isotony. If $I_1 \subseteq I_2$ intervals, then $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$.

Möbius covariance. $U : \text{Möb}(S^1)^{(\infty)} \to \mathbb{U}(\mathcal{H})$ a strongly continuous unitary representation of $\text{Möb}(S^1)^{(\infty)}$ on $\mathcal{H}$ s.t.
$U(\gamma)\mathcal{A}(I)U(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I) \quad \forall\gamma \in \text{Möb}(S^1)^{(\infty)} \; \forall I \in \mathcal{J}$.

Positivity of the energy. The generator H of the rotation subgroup of U is a positive operator on $\mathcal{H}$, called *conformal Hamiltonian*.

Vacuum. A U-invariant vector $\Omega \in \mathcal{H}$, which is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{J}} \mathcal{A}(I)$.

Graded-locality. A self-adjoint $\Gamma \in \mathbb{U}(\mathcal{H})$ s.t. $\Gamma\Omega = \Omega$ and
$\Gamma\mathcal{A}(I)\Gamma = \mathcal{A}(I)$, $\mathcal{A}(I') \subseteq Z\mathcal{A}(I)'Z^*$ for all $I \in \mathcal{J}$ with $Z := \frac{1_{\mathcal{H}} - i\Gamma}{1-i}$.

Diffeomorphism covariance. A strongly continuous projective unitary extension of U to $\mathrm{Diff}^+(S^1)^{(\infty)}$ s.t.:
$U(\gamma)\mathcal{A}(I)U(\gamma)^{-1} = \mathcal{A}(\dot{\gamma}I), \quad \forall\gamma \in \mathrm{Diff}^+(S^1)^{(\infty)}$;
$U(\gamma)AU(\gamma)^{-1} = A, \quad \forall A \in \mathcal{A}(I'), \forall\gamma \in \mathrm{Diff}(I), \forall I \in \mathcal{J}$.

Irreducibility. Ω is the unique vacuum vector up to a phase.

A **vertex operator superalgebra** is a quadruple (V, Ω, Y, ν):

$\mathbb{C}$-vector superspace. $\mathbb{C}$-vector space V with an involution Γ_V s.t.:

$$V_{\bar{0}} := \{a \in V \mid \Gamma_V a = a\}, \quad V_{\bar{1}} := \{a \in V \mid \Gamma_V a = -a\}, \quad V = V_{\bar{0}} \oplus V_{\bar{1}}, \quad \bar{0}, \bar{1} \in \mathbb{Z}/2\mathbb{Z}.$$

$a \in V_{\bar{0}}$ is an **even** element with **parity** $p(a) = \bar{0}$.
$b \in V_{\bar{1}}$ is an **odd** element with **parity** $p(b) = \bar{1}$.

Vacuum vector. $\Omega \in V_{\bar{0}}$.

State-Field correspondence. A $\mathbb{C}$-linear map $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ denoted by the formal series $Y(a, z) := \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ with $a \in V$ s.t.:

- **Parity preserving field**. For every $a, b \in V$, $a_{(n)} b \in V_{p(a)+p(b)}$ for all $n \in \mathbb{Z}$ and $a_{(M)} b = 0$ for $M \gg 0$;
- **Vacuum**. $Y(\Omega, z) = \mathbb{1}_V$ and $a_{(-1)}\Omega = a$ for all $a \in V$;
- **Locality.** For every $a, b \in V$, as formal distribution

$$(z - w)^N [Y(a, z), Y(b, w)] = 0 \qquad N \gg 0 \qquad \text{(all commutators are graded)} \; .$$

Conformal vector. $\nu \in V_{\bar{0}}$, $Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ s.t.:

- **Virasoro algebra cr**. $[L_m, L_n] = (m-n)L_{m+n} + c\frac{(m^3-m)}{12}\delta_{m,-n}1$ with central charge $c \in \mathbb{C}$;
- **Translation covariance**. $L_{-1}\Omega = 0$ and $[L_{-1}, Y(a,z)] = \frac{\mathrm{d}}{\mathrm{d}z}Y(a,z)$ for all $a \in V$;
- $V_{\bar{0}} = \bigoplus_{n \in \mathbb{Z}} V_n$, $V_{\bar{1}} = \bigoplus_{n \in \mathbb{Z}-\frac{1}{2}} V_n$ with $V_n := \mathrm{Ker}(L_0 - n\mathbb{1}_V)$, dim $V_n < +\infty$, $V_n = 0$ for $n \ll 0$.

Nomenclature and notation

- $Y(a,z)$ for $a \in V$ is called **vertex operator**.
- If $a \in V_n$ for some $n \in \frac{1}{2}\mathbb{Z}$, then a is called **homogeneous** of **conformal weight** $d_a := n$. We write

$$Y(a,z) = \sum_{n \in \mathbb{Z} - d_a} a_n z^{-n-d_a}, \qquad a_n := a_{(n+d_a-1)}.$$

- An ideal $\mathscr{J}$ of a vertex operator superalgebra V is an L_{-1}-invariant vector subspace such that $a_{(n)}\mathscr{J} \subseteq \mathscr{J}$ for all $a \in V$ and all $n \in \mathbb{Z}$.
  V is said **simple** if the only ideal are $\{0\}$ and V itself.

An **(anti-)linear automorphism** ϕ on V is an (anti-)linear vector space automorphism s.t. $\phi(\Omega) = \Omega$, $\phi(\nu) = \nu$ and $\phi(a_{(n)}b) = (\phi(a))_{(n)}\phi(b)$ for all $a, b \in V$ and all $n \in \mathbb{Z}$.

A **unitary VOSA** is a VOSA V equipped with:

- a scalar product $(\cdot|\cdot)$, that is, a positive-definite hermitian form (linear in the second variable), which is **normalized**, i.e., $(\Omega|\Omega) = 1$;
- an anti-linear involution θ, called the **PCT operator**;

such that the following **invariant property** holds

$$(Y(\theta(a), z)b|c) = (b|Y(e^{zL_1}(-1)^{2L_0^2+L_0}z^{-2L_0}a, z^{-1})c) \quad \forall a, b, c \in V$$

Constructing graded-local conformal nets

Definition

Given a unitary VOSA V, define the norm $\|\cdot\| := (\cdot|\cdot)^{\frac{1}{2}}$. Then, the separable Hilbert space $\mathcal{H}$ for our graded-local conformal net theory is obtained as the norm completion of V by $\|\cdot\|$.

To construct the local von Neumann algebras:

- the idea is to define some operator-valued distributions from the circle S^1 to $\mathcal{H}$, using vertex operators $Y(a,z)$.

First, we need a control on the operator norm of coefficients a_n:

Definition

A unitary VOSA V is said **energy-bounded** if for every $a \in V$ there exists $k, s, M > 0$ such that

$$\|a_n b\| \leq M(|n|+1)^s \left\|(1_{\mathcal{H}} + L_0)^k b\right\| \qquad \forall n \in \frac{1}{2}\mathbb{Z} \ \forall b \in V\,.$$

Test functions: $C^\infty(S^1)$ and $C^\infty_\chi(S^1) := \chi C^\infty(S^1)$ where $\chi(x) := e^{i\frac{x}{2}}$ with $x \in (-\pi, \pi]$. V energy-bounded unitary VOSA. Define the following operators on V: $a \in V_{\overline{0}}$, $b \in V_{\overline{1}}$, $f \in C^\infty(S^1)$, $g \in C^\infty_\chi(S^1)$, then

$$Y_0(a,f)c := \sum_{n\in\mathbb{Z}} \widehat{f}_n a_n c\,, \qquad Y_0(b,g)c := \sum_{n\in\mathbb{Z}-\frac{1}{2}} \widehat{g}_n b_n c \qquad \forall c \in V\,.$$

- Invariance property $\Rightarrow$ a_n, b_n are closable on $\mathcal{H}$.
- Energy bounds $\Rightarrow$ $Y_0(a,f)$ and $Y_0(b,g)$ are densely defined operator on $\mathcal{H}$.

Definition

Smeared vertex operators: $Y(a,f)$ and $Y(b,g)$ are the closure of $Y_0(a,f)$ and $Y_0(b,g)$ on $\mathcal{H}$.

- Energy bounds (+ some standard results) $\Rightarrow$ for all $c \in \mathcal{H}^\infty$,

$$C^\infty(S^1) \ni f \mapsto Y(a,f)c \in \mathcal{H}^\infty \quad \text{and} \quad C^\infty_\chi(S^1) \ni g \mapsto Y(b,g)c \in \mathcal{H}^\infty$$

are operator-valued distributions ($\mathcal{H}^\infty$ is the common invariant core of smooth vectors for $1_\mathcal{H} + L_0$).

Definition of the net

Let (V, Ω, Y, ν) be a simple energy-bounded VOSA.

- For all $I \in \mathcal{J}$, define the von Neumann algebras

$$\mathcal{A}_{(V,(\cdot|\cdot))}(I) := W^*\left(\left\{ Y(a,f),\ Y(b,g) \mid {\scriptstyle a\in V_{\bar{0}}\,,\ f\in C^\infty(S^1)\,,\ \mathrm{supp} f\subset I \atop b\in V_{\bar{1}}\,,\ g\in C^\infty_\chi(S^1)\,,\ \mathrm{supp} g\subset I} \right\}\right) .$$

- We have a strongly-continuous projective unitary representation $U : \mathrm{Diff}^+(S^1)^{(\infty)} \to \mathbb{U}(\mathcal{H})$ induced by the conformal vector ν, which factors through $\mathrm{Diff}^+(S^1)^{(2)}$:

$$U(\exp^{(2)}(tf))AU(\exp^{(2)}(tf)) = e^{itY(\nu,f)}Ae^{-itY(\nu,f)}$$

  where $\exp^{(2)}(tf)$ is the lift to $\mathrm{Diff}^+(S^1)^{(2)}$ of the exponential map on the real vector field $f\frac{\mathrm{d}}{\mathrm{d}x}$, $f \in C^\infty(S^1, \mathbb{R})$.

Which properties of the net can be proved:

V: simple energy-bounded unitary VOSA;
$(\mathcal{A}_{(V,(\cdot|\cdot))}, U)$: associated family of von Neumann algebras with representation of $\mathrm{Diff}^+(S^1)^{(\infty)}$;

- Isotony ☑;
- Möbius covariance ☑;
- Positivity of the energy: L_0 is the conformal Hamiltonian. ☑;
- $\Omega \in V$ is the vacuum ☑;
- Irreducibility ☑;
- Graded-locality ☒;
- Diffeomorphism covariance ☑ (it is possible to prove it using or without using the graded-locality of the net).

Γ is the extension of Γ_V to $\mathcal{H}$;
$Z := \frac{1_{\mathcal{H}} - i\Gamma}{1-i}$;

Definition

Let V be a unitary VOSA. V is said **strongly graded-local** if it is energy-bounded and $\mathcal{A}_{(V,(\cdot|\cdot))}(I') \subseteq Z\mathcal{A}_{(V,(\cdot|\cdot))}(I)'Z^*$ for all $I \in \mathcal{J}$.

Theorem

Let V be a simple strongly graded-local unitary VOSA, then $\mathcal{A}_{(V,(\cdot|\cdot))}$ is an irreducible graded-local conformal net. Moreover, if $(\cdot|\cdot)'$ determines another unitary structure on V, then $\mathcal{A}_{(V,(\cdot|\cdot))}$ is unitarily isomorphic to $\mathcal{A}_{(V,(\cdot|\cdot)')}$.

Therefore, we indicate the graded-local conformal net so obtained from V with simply $\mathcal{A}_V$.

Further correspondence results

Unitary subalgebras and covariant subnets

V: simple strongly graded-local unitary VOSA.

Theorem

$\mathrm{Aut}_{(\cdot|\cdot)}(V) = \mathrm{Aut}(\mathcal{A}_V)$. In particular, if $\mathrm{Aut}(V)$ is compact, then $\mathrm{Aut}(V) = \mathrm{Aut}(\mathcal{A}_V)$.

Theorem

The map $W \mapsto \mathcal{A}_W$ realises a one-to-one correspondence between unitary subalgebras of V and covariant subnets of $\mathcal{A}_V$.
In particular, such a map "preserves" the coset construction: $\mathcal{A}_{W^c} = \mathcal{A}_W^c$.

Strong graded-locality by generators

$\mathfrak{F}$ a subset of a simple energy-bounded unitary VOSA V. For all $I \in \mathcal{J}$, define

$$\mathcal{A}_{\mathfrak{F}}(I) := W^* \left(\left\{ Y(a,f),\ Y(b,g) \mid {}^{a \in V_{\bar{0}} \cap \mathfrak{F}\,,\ f \in C^\infty(S^1)\,,\ \mathrm{supp} f \subset I}_{b \in V_{\bar{1}} \cap \mathfrak{F}\,,\ g \in C^\infty_\chi(S^1)\,,\ \mathrm{supp} g \subset I} \right\} \right) .$$

Theorem

Assume that $\mathfrak{F}$ contains only quasi-primary vectors ($L_1\mathfrak{F} = \{0\}$) and that it generates V. If there exists an $I \in \mathcal{J}$ such that $\mathcal{A}_{\mathfrak{F}}(I') \subseteq Z\mathcal{A}_{\mathfrak{F}}(I)'Z^*$, then V is strongly graded-local and $\mathcal{A}_{\mathfrak{F}}(I) = \mathcal{A}_V(I)$ for all $I \in \mathcal{J}$.

Corollary

V^1 and V^2 simple strongly graded-local unitary VOSA. Then, $\mathcal{A}_{V^1 \hat{\otimes} V^2} = \mathcal{A}_{V^1} \hat{\otimes} \mathcal{A}_{V^2}$.

Corollary

Let V be a simple unitary VOSA generated by $V_{\frac{1}{2}} \cup V_1 \cup \mathfrak{F}$, where $\mathfrak{F} \subseteq V_2$ is a family of quasi-primary θ-invariant Virasoro vectors. Then V is energy-bounded and strongly graded-local.

Classical examples

We can obtain the following classical examples of graded-local conformal nets through the procedure just described:

- **Real free fermion net**: $\mathfrak{F} := \mathcal{A}_{\mathfrak{F}}$ with $\mathfrak{F}$ the free fermion VOSA.
- **Charged free fermion net**: $\mathfrak{F}^2 := \mathfrak{F}\hat{\otimes}\mathfrak{F} = \mathcal{A}_{F\hat{\otimes}F}$.
- ***d*-fermion net**: $\mathfrak{F}^d = \mathcal{A}_{F^d}$.
- **Lie superalgebra net**: $\mathcal{A}_{\mathfrak{g}_k} \otimes \mathfrak{F}^d$ with $\mathcal{A}_{\mathfrak{g}_k} := \mathcal{A}_{V^k(\mathfrak{g})}$ the net associated to level k Lie algebra $\mathfrak{g}$.
- **Rank-one lattice net**: $\mathcal{A}_N := \mathcal{A}_{V_{L_N}}$ with V_{L_N} the simple unitary VOSA associated to an even/odd one-dimensional lattice $L_N = \sqrt{N}J\mathbb{Z}$.
- $N = 1, 2$ **super-Virasoro net**: it allows us to talk about "**unitary superconformal structures**" on a VOSA.

Theorem

V a simple strongly graded-local unitary VOSA. Then V is superconformal iff $\mathcal{A}_V$ is.

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Thanks for your attention!