Generalized Orbifolds in Conformal Field Theory: Compact Hypergroups

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based on joint work with Marcel Bischoff and Luca Giorgetti

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Let $\mathcal{A}$ be a conformal net and $\mathcal{B} \subset \mathcal{A}$ a conformal subnet.

**Question:** Is there an “algebraic object” $K$ acting on $\mathcal{A}$ for which $\mathcal{B} = \mathcal{A}^K$?

Conversely: If $K$ suitably acts on $\mathcal{A}$, is $\mathcal{B} := \mathcal{A}^K$ a conformal subnet?

Can then have statements of the type:
If $\mathcal{B} \subset \mathcal{A}$ with $\mathcal{B} = \mathcal{A}^K$ then all intermediate conformal subnets $\mathcal{C}$, 

$$\mathcal{B} \subset \mathcal{C} \subset \mathcal{A},$$

are given by fixed points under the action of subobjects of $K$ on $\mathcal{A}$. 

Compact Hypergroups

Definition

We say a compact Hausdorff space $K$ with distinguished element $e \in K$ and continuous involution $\overline{\cdot} : K \to K, k \mapsto \overline{k}$, with $\overline{e} = e$ is a compact hypergroup, if

- The Banach space $\mathcal{M}^b(K)$ of complex Radon measures has a convolution product $\ast$ which is associative, bilinear, weakly continuous.
- For $x, y \in K$ the measure $\delta_x \ast \delta_y$ is a probability measure.
- The mapping $(x, y) \to \delta_x \ast \delta_y$ is continuous.
- $\delta_e \ast \delta_x = \delta_x \ast \delta_e = \delta_x$ for all $x \in K$.
- $\overline{\delta_x \ast \delta_y} = \delta_{\overline{y}} \ast \delta_{\overline{x}}$, where $\overline{\mu(k)} = \mu(\overline{k})$.
- $\exists$ faithful probability measure $\mu_K \in \mathcal{M}^b(K)$ such that for all $f, g \in C(K)$,

$$\int_K f(x \ast y)g(x)d\mu_K(x) = \int_K f(x)g(x \ast \overline{y})d\mu_K(x)$$

with

$$f(x \ast y) := \int_K f(k)d(\delta_x \ast \delta_y)(k).$$
Let $\mathcal{I}$ be the set of nonempty, nondense, open intervals of $S^1$. A conformal net on $S^1$ is a family $\mathcal{A} = \{\mathcal{A}(I) : I \in \mathcal{I}\}$ of von Neumann algebras, acting on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$, satisfying the following properties:

- **Isotony:** $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$.
- **Locality:** $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$, if $I_1 \cap I_2 = \emptyset$, $I_1, I_2 \in \mathcal{I}$.
- **Möbius covariance:** There exists a strongly continuous unitary representation $U$ of $\text{PSL}(2, \mathbb{R})$ in $\mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$, $I \in \mathcal{I}$, $g \in \text{PSL}(2, \mathbb{R})$, where $\text{PSL}(2, \mathbb{R})$ acts on $S^1$ by Möbius transformations.
- **Positivity of the energy:** The generator of rotations $L_0$ has nonnegative spectrum.
- **Existence and uniqueness of the vacuum:** There exists a unique (up to phase) $U$-invariant unit vector $\Omega \in \mathcal{H}$.
- **Cyclicity of the vacuum:** $\Omega$ is cyclic for the algebra $\mathcal{A}(S^1) := \bigcup_{I \in \mathcal{I}} \mathcal{A}(I)$. 
The tensor category \( \text{End}(\mathcal{N}) \)

Let \( \mathcal{A} \) be a conformal net on \( S^1 \), \( \mathcal{B} \subset \mathcal{A} \) a conformal subnet.

Let \( \mathcal{N} := \mathcal{B}(I), \mathcal{M} := \mathcal{A}(I) \) for a fixed \( I \in \mathcal{I} \) (type III factors).

\( \text{End}(\mathcal{N}) \) is the category of unital endomorphisms of \( \mathcal{N} \) with

\[
\text{Hom}(\rho, \sigma) := \{ T \in \mathcal{N} : T\rho(n) = \sigma(n)T \text{ for all } n \in \mathcal{N} \}
\]

for \( \rho, \sigma \in \text{End}(\mathcal{N}) \), and tensor product

\[
\rho \otimes \sigma := \rho \circ \sigma,
\]

\[
T_1 \otimes T_2 := T_1\rho_1(T_2) = \sigma_1(T_2)T_1.
\]

for \( T_1 \in \text{Hom}(\rho_1, \sigma_1), T_2 \in \text{Hom}(\rho_2, \sigma_2), \rho_1, \rho_2, \sigma_1, \sigma_2 \in \text{End}(\mathcal{N}). \)
Longo's Canonical Endomorphism

Let $\mathcal{N} := \mathcal{B}(I)$, $\mathcal{M} := \mathcal{A}(I)$.

Let $\iota: \mathcal{N} \rightarrow \mathcal{M}$ be the embedding homomorphism of $\mathcal{N}$ in $\mathcal{M}$, then

$\gamma := \iota \bar{\iota} \in \text{End}(\mathcal{M})$ is the canonical endomorphism of $\mathcal{N} \subset \mathcal{M}$

$\theta := \bar{\iota} \iota \in \text{End}(\mathcal{N})$ is the dual canonical endomorphism of $\mathcal{N} \subset \mathcal{M}$

- Any $E \in E(\mathcal{M}, \mathcal{N})$ is a Stinespring dilation of $\gamma$: $E(\cdot) = w^* \gamma(\cdot) w$ with $w \in \text{Hom(id, } \theta)$ [Longo 90].

$\gamma$ and $\theta$ may be defined by Tomita-Takesaki modular theory even if $\iota$ does not exist as in a rigid tensor category [Longo 90].

Let $\langle \theta \rangle$ denote the rigid category generated by finite-dimensional subendomorphisms of $\theta$. 

Discreteness

Definition (Discreteness)

If \( \mathcal{N} \subset \mathcal{M} \) is an irreducible subfactor (\( \mathcal{M} \cap \mathcal{N}' = \mathbb{C} \)) with normal, faithful conditional expectation \( E \), then

\[ \mathcal{N} \subset \mathcal{M} \text{ discrete} \iff \theta \cong \bigoplus_i \rho_i \text{ with } \dim(\rho_i) < \infty. \]
Discreteness

\[ \text{Hom}(\theta, \theta) \cong \bigoplus_{[\rho]} M_{n_{\rho}}(\mathbb{C}) \]

**Definition (Charged Intertwiners)**

Let \( \rho \in \text{Obj}(\langle \theta \rangle) \).

\[ H_\rho := \text{Hom}(\iota, \iota \circ \rho) = \{ \psi \in \mathcal{M} : \psi \iota(n) = \iota(\rho(n))\psi \text{ for all } n \in \mathcal{N} \} \]

Recall \( E(\cdot) = w^* \gamma(\cdot)w \) with \( w \in \text{Hom}(\text{id}, \theta) \).

\[ T_\rho : H_\rho \times H_\rho \to \text{Hom}(\theta, \theta) \subset \mathcal{N} \]

\[ (\psi_1, \psi_2) \mapsto \gamma(\psi_1^*)ww^*\gamma(\psi_2) \]
Construction of Hypergroup - Algebra of Trigonometric Polynomials

\[
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\]

**Definition (Algebra of Trigonometric Polynomials)**

\[
\text{Trig}(\mathcal{N} \subset \mathcal{M}) := \text{Span}_\mathbb{C}\{T_{\rho}(\psi_1, \psi_2) : \rho \in \text{Obj}(\langle \theta \rangle), \psi_1, \psi_2 \in \mathcal{H}_{\rho}\},
\]

\[
T_{\rho_1}(\psi_1, \psi_2) \ast T_{\rho_2}(\psi_3, \psi_4) := T_{\rho_1 \circ \rho_2}(\psi_1 \psi_3, \psi_2 \psi_4),
\]

\[
\bullet : \psi \in \mathcal{H}_{\rho} \to \psi^\bullet := \psi^* \nu(\bar{r}_{\rho}) \in \mathcal{H}_{\bar{\rho}},
\]

\[
(T_{\rho}(\psi_1, \psi_2))^\bullet := T_{\bar{\rho}}(\psi_1^\bullet, \psi_2^\bullet).
\]
**Theorem (M. Bischoff, S.D.V, L. Giorgetti)**

\((\text{Trig}(\mathcal{N} \subset \mathcal{M}), *, \cdot)\) is a unital, associative algebra with involution. If \(\mathcal{N} \subset \mathcal{M}\) is a local subfactor then \((\text{Trig}(\mathcal{N} \subset \mathcal{M}), *, \cdot)\) is commutative.

**Definition**

A discrete, irreducible subfactor \(\mathcal{N} \subset \mathcal{M}\) is local if

\[
\psi_\rho \psi_\sigma = \varepsilon_{\rho, \sigma}^\pm \psi_\sigma \psi_\rho
\]

for all \(\psi_\rho \in H_\rho, \psi_\sigma \in H_\sigma, \rho, \sigma \in \text{Obj}(\langle \theta \rangle)\) with \(\varepsilon_{\rho, \sigma}^\pm\) the DHR braiding.
Adjitable UCP maps

Let $\Omega$ be a cyclic and separating vector for $\mathcal{M}$.

**Definition**

Let $\text{UCP}_\mathcal{N}(\mathcal{M}, \Omega)$ denote the set of normal linear maps $\phi: \mathcal{M} \rightarrow \mathcal{M}$ with the following properties

1. $\phi$ is unital and completely positive (UCP).
2. $\phi$ preserves the state given by $\Omega$, i.e. $(\Omega, \phi(\cdot)\Omega) = (\Omega, \cdot \Omega)$.
3. $\phi$ is $\mathcal{N}$-bimodular, i.e. $\phi(n_1 mn_2) = n_1 \phi(m)n_2$ for every $n_1, n_2 \in \mathcal{N}$, $m \in \mathcal{M}$.

**Lemma**

*Under our hypothesis for $\mathcal{N} \subset \mathcal{M}$, every $\phi \in \text{UCP}_\mathcal{N}(\mathcal{M}, \Omega)$ is $\Omega$-adjointable, i.e. $\exists \phi^\# \in \text{UCP}_\mathcal{N}(\mathcal{M}, \Omega)$ such that*

$$(\phi^\#(m_1)\Omega, m_2\Omega) = (m_1\Omega, \phi(m_2)\Omega)$$

*for every $m_1, m_2 \in \mathcal{M}$.*
There is a duality pairing between $\text{UCP}_\mathcal{N}(\mathcal{M}, \Omega)$ and $(\text{Trig}(\mathcal{N} \subset \mathcal{M}), \ast, \bullet)$

$$\langle \cdot, \cdot \rangle : \text{UCP}_\mathcal{N}(\mathcal{M}, \Omega) \times \text{Trig}(\mathcal{N} \subset \mathcal{M}) \to \mathbb{C}$$

$$\langle \phi, T_\rho(\psi_1, \psi_2) \rangle := \psi_1^\ast \phi(\psi_2) \in \text{Hom}(\nu, \nu) = \mathbb{C} \text{id}$$

Denote by $\omega_\phi := \langle \phi, \cdot \rangle$. $\omega_\phi$ is a positive linear functional.
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$$\langle \phi, T_\rho(\psi_1, \psi_2) \rangle := \psi_1^* \phi(\psi_2) \in \text{Hom}(\nu, \nu) = \mathbb{C} \text{id}$$

Denote by $\omega_\phi := \langle \phi, \cdot \rangle$. $\omega_\phi$ is a positive linear functional.

Consider $\omega_E$, with $E$ conditional expectation of $\mathcal{N} \subset \mathcal{M}$. Let $\lambda_E$ be the GNS representation of $(\text{Trig}(\mathcal{N} \subset \mathcal{M}), \ast, \bullet)$ wrt $\omega_E$.

**Lemma**

$\lambda_E$ is a faithful representation of $(\text{Trig}(\mathcal{N} \subset \mathcal{M}), \ast, \bullet)$ by bounded operators.
Reduced $C^*$-algebra of $\mathcal{N} \subset \mathcal{M}$

Let $\lambda_E$ be the GNS representation of $(\text{Trig}(\mathcal{N} \subset \mathcal{M}), \ast, \cdot)$ wrt $\omega_E$.

Denote by $C^*_E(\mathcal{N} \subset \mathcal{M})$ the closure of the image of $\lambda_E$.

**Remark**

$C^*_E(\mathcal{N} \subset \mathcal{M})$ is a commutative and separable $C^*$-algebra and thus

$$C^*_E(\mathcal{N} \subset \mathcal{M}) \cong C(K)$$

for some compact, metrizable topological space $K$. 
Reduced $C^*$-algebra of $\mathcal{N} \subset \mathcal{M}$

Let $\lambda_{E}$ be the GNS representation of $(\text{Trig}(\mathcal{N} \subset \mathcal{M}), \ast, \cdot)$ wrt $\omega_{E}$.

Denote by $C_{E}^{\ast}(\mathcal{N} \subset \mathcal{M})$ the closure of the image of $\lambda_{E}$.

**Remark**

$C_{E}^{\ast}(\mathcal{N} \subset \mathcal{M})$ is a commutative and separable $C^*$-algebra and thus

$$C_{E}^{\ast}(\mathcal{N} \subset \mathcal{M}) \cong C(K)$$

for some compact, metrizable topological space $K$.

**Theorem**

The duality map $\phi \rightarrow \omega_{\phi}$ lifts to a map between $UCP_{\mathcal{N}}(\mathcal{M}, \Omega)$ and states on $C_{E}^{\ast}(\mathcal{N} \subset \mathcal{M})$. 
The Hypergroup $K$

\[ C^*_E(N \subset M) \cong C(K) \]

for some compact, metrizable topological space $K$.

**Theorem (M. Bischoff. S.D.V., L. Giorgetti)**

*The duality map*

\[ \phi \mapsto \omega_{\phi} \]

*is a homeomorphism between*

- \( UCP_N(M, \Omega) \)
- *the set of states* \( \omega \) *on* \( C^*_E(N \subset M) \).

*In particular it restricts to a homeomorphism between*

- \( \text{Extr}(UCP_N(M, \Omega)) \) *(*Extreme points of* \( UCP_N(M, \Omega) \))
- \( K \).
The Hypergroup $K$

**Corollary**

$$UCP_N(\mathcal{M}, \Omega) \cong M^b_+(\text{Extr}(UCP_N(\mathcal{M}, \Omega)))$$

where $M^b_+(\text{Extr}(UCP_N(\mathcal{M}, \Omega)))$ denotes the positive Radon measures on $\text{Extr}(UCP_N(\mathcal{M}, \Omega))$.

**Theorem (M. Bischoff. S.D.V., L. Giorgetti)**

$K \cong \text{Extr}(UCP_N(\mathcal{M}, \Omega))$ is a compact hypergroup with

- **convolution**: $\phi_1 \ast \phi_2 := \phi_1 \circ \phi_2$
- **adjoint**: $\phi^\sim := \phi^\#$, where $\phi^\#$ is the $\Omega$-adjoint of $\phi$.
- $E$ is the Haar measure of $K$. 

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Corollary (Choquet-type decomposition of $E$)

Let $m \in \mathcal{M}$, we have

$$E(m) = \int_K \phi(m) d\omega_E(\phi)$$

(1)

where the integral is understood in the weak sense.

Theorem (M. Bischoff. S.D.V., L. Giorgetti)

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible local discrete subfactor of type III. Let $\Omega \in \mathcal{H}$ be a standard vector for $\mathcal{M} \subset \mathcal{B(H)}$ such that the associated state is invariant wrt the unique normal faithful conditional expectation $E : \mathcal{M} \to \mathcal{N}$.

Then there is a compact hypergroup $K$ which acts on $\mathcal{M}$ by $\Omega$-Markov maps and

$$\mathcal{N} = \mathcal{M}^K := \{ m \in \mathcal{M} : \phi(m) = m \text{ for all } \phi \in K \}.$$
Proposition

There is a bijective correspondence between

\[ \{ \pi : \text{Irreducible Representations of } K \} \longleftrightarrow \{ \rho_\pi : \text{Irreducible subsectors of } \theta \} \]
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Theorem (R. Vrem 79, Y. Chapovski, L. Vainerman 99 - Schur’s Orthogonality Relations)

Let \( \pi \) be an irreducible representation of \( K \) acting on the complex Hilbert space \( \mathcal{H}_\pi \). There is \( d_\pi \in \mathbb{R} \) with \( d_\pi \geq \dim \mathcal{H}_\pi \) such that for any \( v \in \mathcal{H}_\pi \) with \( \|v\| = 1 \)

\[
\int_K |(v, \pi(k)v)|^2 d\mu_K(k) = \frac{1}{d_\pi}.
\]

\( d_\pi \) is called the hyperdimension of \( \pi \).
Representations

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\]

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Proposition

The hyperdimension \( d_\pi \) is equal to the statistical dimension \( d_{\rho_\pi} \).
The Case of Depth 2

Corollary

\[ \text{Hom}(\gamma, \gamma) \cong L^\infty(K, d\omega_E). \]

together with

Theorem (M. Enock, R. Nest 96)

Suppose \( N \subset M \) has depth 2, is semidiscrete, and \( \text{Hom}(\gamma, \gamma) \) is commutative. Then \( \exists G \) compact group acting on \( M \) with \( N = MG \).

we get

Corollary

Suppose \( N \subset M \) is a discrete local subfactor with depth 2. Then \( \exists G \) compact group acting on \( M \) with \( N = MG \).
Proposition

Let $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ for some compact group $G$. Let $H \subset G$ a closed subgroup. Then

$$\mathcal{N} = (\mathcal{M}^H)^{G//H} \subset \mathcal{M}^H$$

where

$$G//H = \{HxH : x \in G\},$$

$$\delta_{HxH} \ast \delta_{HyH} := \int_H \delta_{HxtyH} d\omega_H(t),$$

$$(\delta_{HxH})^\sim := \delta_{Hx^{-1}H}.$$
Example

\[ \text{Vir}_1 = SU(2)_{1}^{SO(3)} \subset SU(2)_1. \]

Every discrete extension of \( \text{Vir}_1 \) is of the form

\[ \text{Vir}_1 \subset SU(2)_1^H \]

for some closed subgroup \( H \) of \( SO(3) \).

Thus for any discrete extension \( B \) of \( \text{Vir}_1 \),

\[ \text{Vir}_1 = B^K \]

where \( K = SO(3) \parallel H \), with \( H \) a closed subgroup of \( SO(3) \).
Example

\[ SU(2)_{10} \subset Spin(5)_1 \]
\[ \gamma = \text{id} \oplus \gamma_1, \quad d_{\gamma_1} = 2 + \sqrt{3} \]

\[ K = \{ e, \phi_{\gamma_1} \} \]
\[ \phi_{\gamma_1} \ast \phi_{\gamma_1} = \frac{1}{2 + \sqrt{3}} e + \frac{1 + \sqrt{3}}{2 + \sqrt{3}} \phi_{\gamma_1} \]
\[ \phi^\#_{\gamma_1} = \phi_{\gamma_1}. \]
Open Problems and Further Directions

- Inverse Problem: From action of compact hypergroup to discrete subnet?
- Galois type correspondence for Intermediate subnets?
- Non Local case $\leftrightarrow$ Compact Quantum Hypergroups?
- General case (semidiscreteness)?