Generalized Orbifolds in Conformal Field Theory: Compact Hypergroups

Simone Del Vecchio

based on joint work with Marcel Bischoff and Luca Giorgetti

Virtual LQP / LQP45 workshop

17 June 2020

Let $\mathcal{A}$ be a conformal net and $\mathcal{B} \subset \mathcal{A}$ a conformal subnet.

Question: Is there an "algebraic object" K acting on $\mathcal{A}$ for which $\mathcal{B} = \mathcal{A}^K$?

Conversely: If K suitably acts on $\mathcal{A}$, is $\mathcal{B} := \mathcal{A}^K$ a conformal subnet?

Can then have statements of the type:
If $\mathcal{B} \subset \mathcal{A}$ with $\mathcal{B} = \mathcal{A}^K$ then all intermediate conformal subnets $\mathcal{C}$,

$$\mathcal{B} \subset \mathcal{C} \subset \mathcal{A},$$

are given by fixed points under the action of subobjects of K on $\mathcal{A}$.

Definition

We say a compact Hausdorff space K with distinguished element $e \in K$ and continuous involution $\bar{\cdot}: K \to K, k \mapsto \bar{k}$, with $\bar{e} = e$ is a compact hypergroup, if

- The Banach space $M^b(K)$ of complex Radon measures has a convolution product $*$ which is associative, bilinear, weakly continuous.
- For $x, y \in K$ the measure $\delta_x * \delta_y$ is a probability measure.
- The mapping $(x, y) \to \delta_x * \delta_y$ is continuous.
- $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
- $\overline{\delta_x * \delta_y} = \delta_{\bar{y}} * \delta_{\bar{x}}$, where $\overline{\mu}(k) = \mu(\bar{k})$.
- $\exists$ faithful probability measure $\mu_K \in M^b(K)$ such that for all $f, g \in C(K)$,

  $$\int_K f(x * y) g(x) d\mu_K(x) = \int_K f(x) g(x * \bar{y}) d\mu_K(x)$$

  with

  $$f(x * y) := \int_K f(k) d(\delta_x * \delta_y)(k).$$

Let $\mathcal{I}$ be the set of nonempty, nondense, open intervals of S^1. A conformal net on S^1 is a family $\mathcal{A} = \{\mathcal{A}(I) : I \in \mathcal{I}\}$ of von Neumann algebras, acting on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$, satisfying the following properties:

- Isotony: $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2, I_1, I_2 \in \mathcal{I}$.
- Locality: $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$, if $I_1 \cap I_2 = \emptyset, I_1, I_2 \in \mathcal{I}$.
- Möbius covariance: There exists a strongly continuous unitary representation U of PSL$(2, \mathbb{R})$ in $\mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = A(gI)$, $I \in \mathcal{I}$, $g \in$PSL$(2, \mathbb{R})$, where PSL$(2, \mathbb{R})$ acts on S^1 by Möbius transformations.
- Positivity of the energy: The generator of rotations L_0 has nonnegative spectrum.
- Existence and uniqueness of the vacuum: There exists a unique (up to phase) U-invariant unit vector $\Omega \in \mathcal{H}$.
- Cyclicity of the vacuum: Ω is cyclic for the algebra $\mathcal{A}(S^1) := \cup_{I \in \mathcal{I}} \mathcal{A}(I)$.

Let $\mathcal{A}$ be a conformal net on S^1, $\mathcal{B} \subset \mathcal{A}$ a conformal subnet.
Let $\mathcal{N} := \mathcal{B}(I)$, $\mathcal{M} := \mathcal{A}(I)$ for a fixed $I \in \mathcal{I}$ (type III factors).

End($\mathcal{N}$) is the category of unital endomorphisms of $\mathcal{N}$ with

$$\mathrm{Hom}(\rho, \sigma) := \{T \in \mathcal{N} : T\rho(n) = \sigma(n)T \text{ for all } n \in \mathcal{N}\}$$

for $\rho, \sigma \in$End($\mathcal{N}$), and tensor product

$$\rho \otimes \sigma := \rho \circ \sigma,$$

$$T_1 \otimes T_2 := T_1\rho_1(T_2) = \sigma_1(T_2)T_1.$$

for $T_1 \in \mathrm{Hom}(\rho_1, \sigma_1)$, $T_2 \in \mathrm{Hom}(\rho_2, \sigma_2)$, $\rho_1, \rho_2, \sigma_1, \sigma_2 \in$End($\mathcal{N}$).

Let $\mathcal{N} := \mathcal{B}(I)$, $\mathcal{M} := \mathcal{A}(I)$.

Let $\iota : \mathcal{N} \to \mathcal{M}$ be the embedding homomorphism of $\mathcal{N}$ in $\mathcal{M}$, then

$\gamma := \iota\bar{\iota} \in \mathrm{End}(\mathcal{M})$ is the canonical endomorphism of $\mathcal{N} \subset \mathcal{M}$

$\theta := \bar{\iota}\iota \in \mathrm{End}(\mathcal{N})$ is the dual canonical endomorphism of $\mathcal{N} \subset \mathcal{M}$

- Any $E \in E(\mathcal{M}, \mathcal{N})$ is a Stinespring dilation of γ: $E(\cdot) = w^*\gamma(\cdot)w$ with $w \in \mathrm{Hom}(\mathrm{id}, \theta)$ [Longo 90].

γ and θ may be defined by Tomita-Takesaki modular theory even if $\bar{\iota}$ does not exist as in a rigid tensor category [Longo 90].

Let $\langle\theta\rangle$ denote the rigid category generated by finite-dimensional subendomorphisms of θ.

Definition (Discreteness)

If $\mathcal{N} \subset \mathcal{M}$ is an irreducible subfactor ($\mathcal{M} \cap \mathcal{N}' = \mathbb{C}$) with normal, faithful conditional expectation E, then

$\mathcal{N} \subset \mathcal{M}$ discrete $\Leftrightarrow \theta \cong \oplus_i \rho_i$ with $\dim(\rho_i) < \infty$.

$$\mathrm{Hom}(\theta,\theta) \cong \bigoplus_{[\rho]} M_{n_\rho}(\mathbb{C})$$

Definition (Charged Intertwiners)

Let $\rho \in \mathsf{Obj}(\langle\theta\rangle)$.

$$H_\rho := \mathrm{Hom}(\iota, \iota\circ\rho) = \{\psi \in \mathcal{M} : \psi\iota(n) = \iota(\rho(n))\psi \text{ for all } n \in \mathcal{N}\}.$$

Recall $E(\cdot) = w^*\gamma(\cdot)w$ with $w \in \mathrm{Hom}(\mathrm{id},\theta)$.

$$\begin{aligned} T_\rho \colon H_\rho \times H_\rho &\to \mathrm{Hom}(\theta,\theta) \subset \mathcal{N} \\ (\psi_1,\psi_2) &\mapsto \gamma(\psi_1^*)ww^*\gamma(\psi_2) \end{aligned}$$

$$\mathrm{Hom}(\theta, \theta) \cong \bigoplus_{[\rho]} M_{n_\rho}(\mathbb{C})$$

Recall $E(\cdot) = w^* \gamma(\cdot) w$ with $w \in \mathrm{Hom}(\mathrm{id}, \theta)$.

$$\begin{aligned} T_\rho \colon H_\rho \times H_\rho &\to \mathrm{Hom}(\theta, \theta) \subset \mathcal{N} \\ (\psi_1, \psi_2) &\mapsto \gamma(\psi_1^*) w w^* \gamma(\psi_2) \end{aligned}$$

Definition (Algebra of Trigonometric Polynomials)

$$\mathsf{Trig}(\mathcal{N} \subset \mathcal{M}) := \mathsf{Span}_{\mathbb{C}}\{T_\rho(\psi_1, \psi_2) : \rho \in \mathsf{Obj}(\langle\theta\rangle), \psi_1, \psi_2 \in \mathcal{H}_\rho\},$$

$$T_{\rho_1}(\psi_1, \psi_2) * T_{\rho_2}(\psi_3, \psi_4) := T_{\rho_1 \circ \rho_2}(\psi_1 \psi_3, \psi_2 \psi_4),$$

$$\bullet : \psi \in \mathcal{H}_\rho \to \psi^\bullet := \psi^* \iota(\bar{r}_\rho) \in \mathcal{H}_{\bar{\rho}},$$

$$(T_\rho(\psi_1, \psi_2))^\bullet := T_{\bar{\rho}}(\psi_1^\bullet, \psi_2^\bullet).$$

Theorem (M. Bischoff, S.D.V, L. Giorgetti)

$(\mathit{Trig}(\mathcal{N} \subset \mathcal{M}), *, ^{\bullet})$ is a unital, associative algebra with involution.
If $\mathcal{N} \subset \mathcal{M}$ is a local subfactor then $(\mathit{Trig}(\mathcal{N} \subset \mathcal{M}), *, ^{\bullet})$ is commutative.

Definition

A discrete, irreducible subfactor $\mathcal{N} \subset \mathcal{M}$ is local if

$$\psi_\rho \psi_\sigma = \varepsilon^{\pm}_{\rho,\sigma} \psi_\sigma \psi_\rho$$

for all $\psi_\rho \in H_\rho, \psi_\sigma \in H_\sigma$, $\rho, \sigma \in$Obj$(\langle\theta\rangle)$ with $\varepsilon^{\pm}_{\rho,\sigma}$ the DHR braiding.

Let Ω be a cyclic and separating vector for $\mathcal{M}$.

Definition

Let $\mathrm{UCP}_{\mathcal{N}}(\mathcal{M}, \Omega)$ denote the set of normal linear maps $\phi\colon \mathcal{M} \to \mathcal{M}$ with the following properties

1. ϕ is unital and completely positive (UCP).
2. ϕ preserves the state given by Ω, i.e. $(\Omega, \phi(\,\cdot\,)\Omega) = (\Omega, \cdot\,\Omega)$.
3. ϕ is $\mathcal{N}$-bimodular, i.e. $\phi(n_1 m n_2) = n_1 \phi(m) n_2$ for every $n_1, n_2 \in \mathcal{N}$, $m \in \mathcal{M}$.

Lemma

Under our hypothesis for $\mathcal{N} \subset \mathcal{M}$, every $\phi \in$UCP$_{\mathcal{N}}(\mathcal{M}, \Omega)$ is Ω-adjointable, i.e. $\exists \phi^{\#} \in$UCP$_{\mathcal{N}}(\mathcal{M}, \Omega)$ such that

$$(\phi^{\#}(m_1)\Omega, m_2\Omega) = (m_1\Omega, \phi(m_2)\Omega)$$

for every $m_1, m_2 \in \mathcal{M}$.

There is a duality pairing between $\mathsf{UCP}_{\mathcal{N}}(\mathcal{M}, \Omega)$ and $(\mathsf{Trig}(\mathcal{N} \subset \mathcal{M}), *, ^{\bullet})$

$$\langle \cdot , \cdot \rangle \colon \mathsf{UCP}_{\mathcal{N}}(\mathcal{M}, \Omega) \times \mathsf{Trig}(\mathcal{N} \subset \mathcal{M}) \to \mathbb{C}$$

$$\langle \phi, T_\rho(\psi_1, \psi_2) \rangle := \psi_1^* \phi(\psi_2) \in \mathrm{Hom}(\iota, \iota) = \mathbb{C}\, \mathrm{id}$$

Denote by $\omega_\phi := \langle \phi, \cdot \rangle$. ω_ϕ is a positive linear functional.

There is a duality pairing between $\mathsf{UCP}_{\mathcal{N}}(\mathcal{M},\Omega)$ and $(\mathsf{Trig}(\mathcal{N}\subset\mathcal{M}),*,^\bullet)$

$$\langle\cdot\,,\,\cdot\rangle\colon \mathsf{UCP}_{\mathcal{N}}(\mathcal{M},\Omega)\times\mathsf{Trig}(\mathcal{N}\subset\mathcal{M})\to\mathbb{C}$$
$$\langle\phi,T_\rho(\psi_1,\psi_2)\rangle:=\psi_1^*\phi(\psi_2)\in\mathrm{Hom}(\iota,\iota)=\mathbb{C}\,\mathrm{id}$$

Denote by $\omega_\phi:=\langle\phi,\cdot\rangle$. ω_ϕ is a positive linear functional.

Consider ω_E, with E conditional expectation of $\mathcal{N}\subset\mathcal{M}$.
Let λ_E be the GNS representation of $(\mathsf{Trig}(\mathcal{N}\subset\mathcal{M}),*,^\bullet)$ wrt ω_E.

Lemma

λ_E is a faithful representation of $(\mathsf{Trig}(\mathcal{N}\subset\mathcal{M}),*,^\bullet)$ by bounded operators.

Reduced C^*-algebra of $\mathcal{N} \subset \mathcal{M}$

Let λ_E be the GNS representation of $(\mathrm{Trig}(\mathcal{N} \subset \mathcal{M}), *, ^\bullet)$ wrt ω_E.

Denote by $C^*_E(\mathcal{N} \subset \mathcal{M})$ the closure of the image of λ_E.

Remark

$C^*_E(\mathcal{N} \subset \mathcal{M})$ is a commutative and separable C^*-algebra and thus

$$C^*_E(\mathcal{N} \subset \mathcal{M}) \cong C(K)$$

for some compact, metrizable topological space K.

Let λ_E be the GNS representation of $(\mathrm{Trig}(\mathcal{N} \subset \mathcal{M}), *, ^\bullet)$ wrt ω_E.

Denote by $C^*_E(\mathcal{N} \subset \mathcal{M})$ the closure of the image of λ_E.

Remark

$C^*_E(\mathcal{N} \subset \mathcal{M})$ is a commutative and separable C^*-algebra and thus

$$C^*_E(\mathcal{N} \subset \mathcal{M}) \cong C(K)$$

for some compact, metrizable topological space K.

Theorem

The duality map $\phi \to \omega_\phi$ lifts to a map between UCP$_\mathcal{N}(\mathcal{M}, \Omega)$ and states on $C^*_E(\mathcal{N} \subset \mathcal{M})$.

$$C^*_E(\mathcal{N} \subset \mathcal{M}) \cong C(K)$$

for some compact, metrizable topological space K.

Theorem (M. Bischoff. S.D.V., L. Giorgetti)

The duality map

$$\phi \mapsto \omega_\phi$$

is a homeomorphism between

- $UCP_{\mathcal{N}}(\mathcal{M}, \Omega)$
- *the set of states* ω *on* $C^*_E(\mathcal{N} \subset \mathcal{M})$.

In particular it restricts to a homeomorphism between

- $Extr(UCP_{\mathcal{N}}(\mathcal{M}, \Omega))$ *(Extreme points of* $UCP_{\mathcal{N}}(\mathcal{M}, \Omega)$*)*
- K.

Corollary

$$\mathit{UCP}_{\mathcal{N}}(\mathcal{M},\Omega) \cong M^b_+(\mathit{Extr}(\mathit{UCP}_{\mathcal{N}}(\mathcal{M},\Omega)))$$

where $M^b_+(\mathit{Extr}(\mathit{UCP}_{\mathcal{N}}(\mathcal{M},\Omega)))$ *denotes the positive Radon measures on* $\mathit{Extr}(\mathit{UCP}_{\mathcal{N}}(\mathcal{M},\Omega))$.

Theorem (M. Bischoff. S.D.V., L. Giorgetti)

$K \cong \mathit{Extr}(\mathit{UCP}_{\mathcal{N}}(\mathcal{M},\Omega))$ *is a compact hypergroup with*

- *convolution:* $\phi_1 * \phi_2 := \phi_1 \circ \phi_2$
- *adjoint:* $\phi^\sim := \phi^\#$, *where* $\phi^\#$ *is the* Ω*-adjoint of* ϕ.
- E *is the Haar measure of* K.

Corollary (Choquet-type decomposition of E)

Let $m \in \mathcal{M}$, we have

$$E(m) = \int_K \phi(m) d\omega_E(\phi) \tag{1}$$

where the integral is understood in the weak sense.

Theorem (M. Bischoff. S.D.V., L. Giorgetti)

Let $\mathcal{N} \subset \mathcal{M}$ be an irreducible local discrete subfactor of type III. Let $\Omega \in \mathcal{H}$ be a standard vector for $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ such that the associated state is invariant wrt the unique normal faithful conditional expectation $E : \mathcal{M} \to \mathcal{N}$.
Then there is a compact hypergroup K which acts on $\mathcal{M}$ by Ω-Markov maps and

$$\mathcal{N} = \mathcal{M}^K := \{m \in \mathcal{M} : \phi(m) = m \text{ for all } \phi \in K\}.$$

Proposition

There is a bijective correspondence between

$$\{\pi : \textit{ Irreducible Representations of } K\ \} \longleftrightarrow \{\rho_\pi : \textit{ Irreducible subsectors of } \theta\ \}$$

Proposition

There is a bijective correspondence between

$$\{\pi : \textit{ Irreducible Representations of } K \} \longleftrightarrow \{\rho_\pi : \textit{ Irreducible subsectors of } \theta \}$$

Theorem (R. Vrem 79, Y. Chapovski, L. Vainerman 99 - Schur's Orthogonality Relations)

Let π be an irreducible representation of K acting on the complex Hilbert space $\mathcal{H}_\pi$. There is $d_\pi \in \mathbb{R}$ with $d_\pi \geq$dim$\mathcal{H}_\pi$ such that for any $v \in \mathcal{H}_\pi$ with $\|v\| = 1$

$$\int_K |(v, \pi(k)v)|^2 d\mu_K(k) = \frac{1}{d_\pi}.$$

d_π is called the hyperdimension of π.

Proposition

There is a bijective correspondence between

$$\{\pi : \textit{ Irreducible Representations of } K\ \} \longleftrightarrow \{\rho_\pi : \textit{ Irreducible subsectors of } \theta\ \}$$

Theorem (R. Vrem 79, Y. Chapovski, L. Vainerman 99 - Schur's Orthogonality Relations)

Let π be an irreducible representation of K acting on the complex Hilbert space $\mathcal{H}_\pi$. There is $d_\pi \in \mathbb{R}$ with $d_\pi \geq \dim \mathcal{H}_\pi$ such that for any $v \in \mathcal{H}_\pi$ with $\|v\| = 1$

$$\int_K |(v, \pi(k)v)|^2 d\mu_K(k) = \frac{1}{d_\pi}.$$

d_π is called the hyperdimension of π.

Proposition

The hyperdimension d_π is equal to the statistical dimension d_{ρ_π}.

Corollary

$$\mathrm{Hom}(\gamma, \gamma) \cong L^\infty(K, d\omega_E).$$

together with

Theorem (M. Enock, R. Nest 96)

Suppose $\mathcal{N} \subset \mathcal{M}$ has depth 2, is semidiscrete, and $\mathrm{Hom}(\gamma, \gamma)$ is commutative. Then $\exists G$ compact group acting on $\mathcal{M}$ with $\mathcal{N} = \mathcal{M}^G$.

we get

Corollary

Suppose $\mathcal{N} \subset \mathcal{M}$ is a discrete local subfactor with depth 2. Then $\exists G$ compact group acting on $\mathcal{M}$ with $\mathcal{N} = \mathcal{M}^G$.

Proposition

Let $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ for some compact group G.
Let $H \subset G$ a closed subgroup. Then

$$\mathcal{N} = (\mathcal{M}^H)^{G /\!/ H} \subset \mathcal{M}^H$$

where

$$G /\!/ H = \{HxH : x \in G\},$$

$$\delta_{HxH} * \delta_{HyH} := \int_H \delta_{HxtyH} d\omega_H(t),$$

$$(\delta_{HxH})^\sim := \delta_{Hx^{-1}H}.$$

Example

$$\mathsf{Vir}_1 = SU(2)_1^{SO(3)} \subset SU(2)_1.$$

Every discrete extension of Vir_1 is of the form

$$\mathsf{Vir}_1 \subset SU(2)_1^H$$

for some closed subgroup H of $SO(3)$.

Thus for any discrete extension $\mathcal{B}$ of Vir_1,

$$\mathsf{Vir}_1 = \mathcal{B}^K$$

where $K = SO(3)/\!\!/H$, with H a closed subgroup of $SO(3)$.

Example

$$SU(2)_{10} \subset \mathsf{Spin}(5)_1$$

$$\gamma = \mathrm{id} \oplus \gamma_1, \quad d_{\gamma_1} = 2 + \sqrt{3}$$

$$K = \{e, \phi_{\gamma_1}\}$$

$$\phi_{\gamma_1} * \phi_{\gamma_1} = \frac{1}{2+\sqrt{3}} e + \frac{1+\sqrt{3}}{2+\sqrt{3}} \phi_{\gamma_1}$$

$$\phi_{\gamma_1}^{\#} = \phi_{\gamma_1}.$$

- Inverse Problem: From action of compact hypergroup to discrete subnet?
- Galois type correspondence for Intermediate subnets?
- Non Local case $\longleftrightarrow$ Compact Quantum Hypergroups?
- General case (semidiscreteness)?