Dynamics and fields for tensor networks

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Quantum phases of a chain of strongly interacting anyons

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Quantum gates for the manipulation of topological qubits rely on interactions between non-Abelian anyonic quasiparticles. We study the collective behaviour of systems of anyons arising from such interactions. In particular, we study the effect of favouring different fusion channels of the screened Majorana spins appearing in the recently proposed topological Kondo effect. Based on the numerical solution of a chain of $SO(5)_2$ anyons we identify two critical phases whose low-energy behaviour is characterised by conformal field theories with central charges $c = 1$ and $c = 8/7$, respectively. Our results are complemented by exact results for special values of the coupling constants which provide additional information about the corresponding phase transitions.

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Low-dimensional quantum systems hold an irresistible and enduring fascination because they can support topological states of matter with exotic quasiparticles, *anyons*, exhibiting unusual braiding statistics [1]. While initially a curiosity, anyons generated considerable excitement when it was realized that the fractional quantum Hall effect [2] — and later nanowires [3, 4] and the p_x+ip_y

time history in order to discuss their dynamics. While the non-interacting case is now becoming well understood (see, e.g., [10]) the classification of phases for systems of *interacting* anyons has progressed much slower. An additional complication is that the description of the dynamics of a highly entangled $SO(M)$ Majorana spin in the topological Kondo model, and the collective behaviour

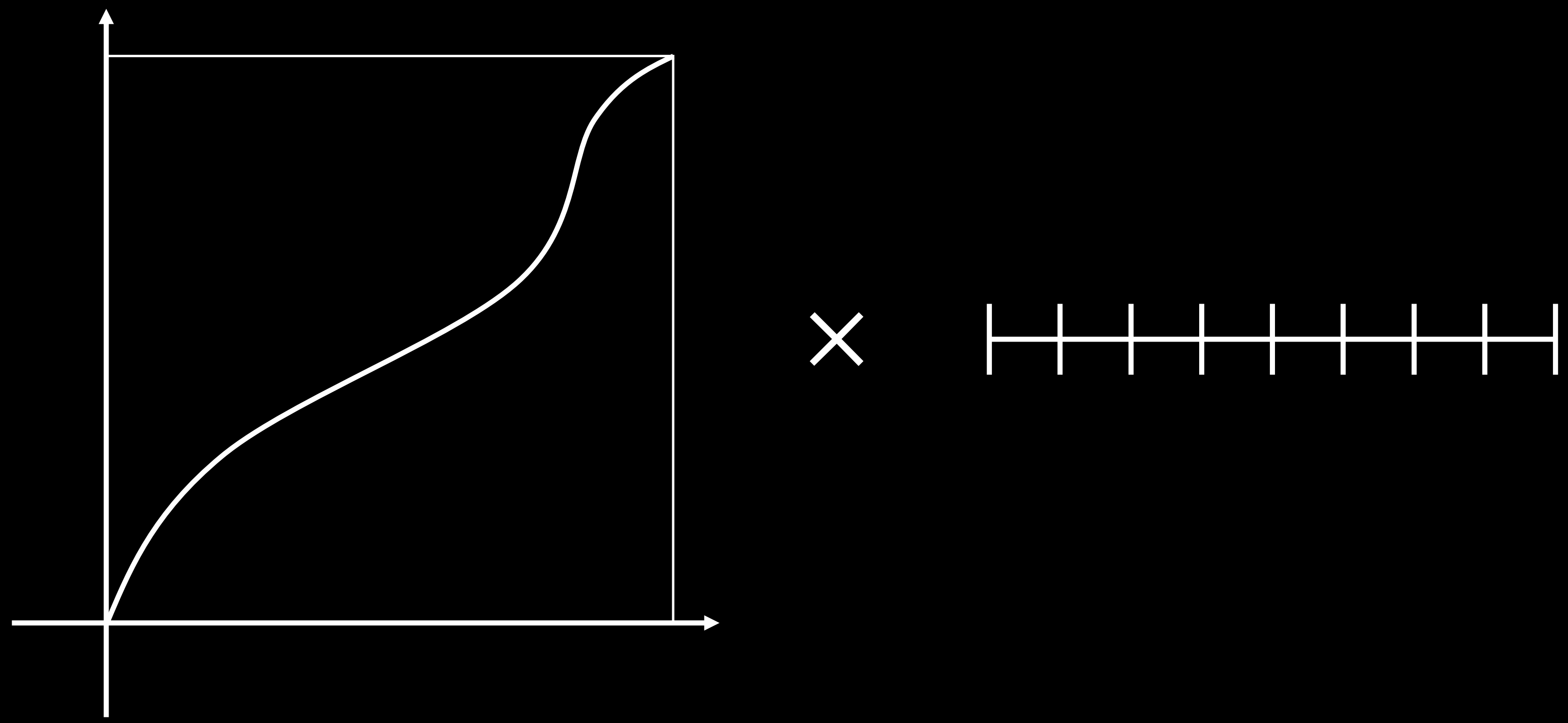
P. E. Finch, H. Frahm, M. Lewerenz, A. Milsted, TJO, Phys. Rev. B 90, 081111 (2014)

?

CFT Dream:
find unitary action of conformal group on low-energy space of quantum spin system

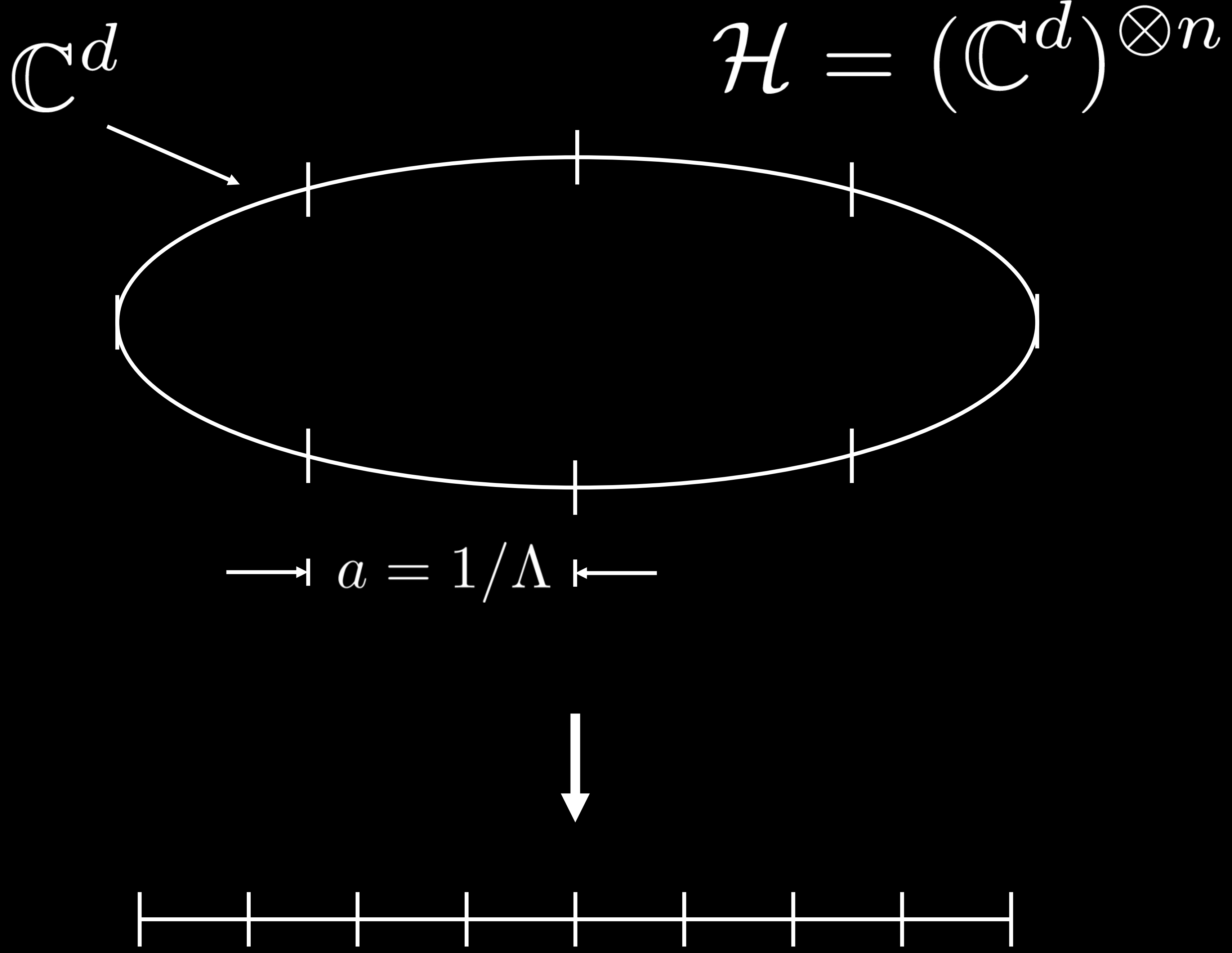
Conformal group:

$$\mathrm{conf}(\mathbb{R}^{1,1}) \cong \mathrm{diff}_+(S^1) \times \mathrm{diff}_+(S^1)$$



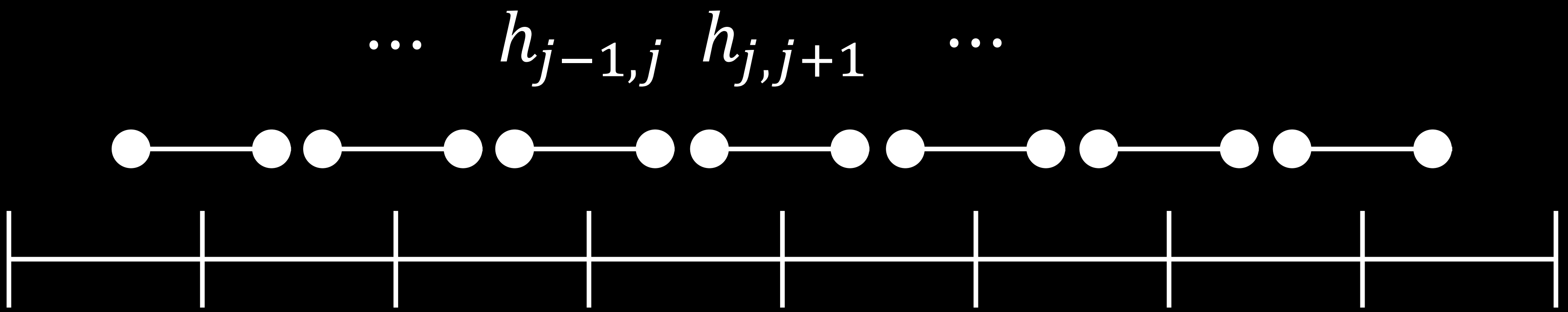
MAIN TASK:
find TNS subspaces for
low energy & **large scale**
excitations which admit
"conformal group action"

Kadanoff bock spin renormalisation


$\mathbb{C}^d$
$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$
$a = 1/\Lambda$

This is important!

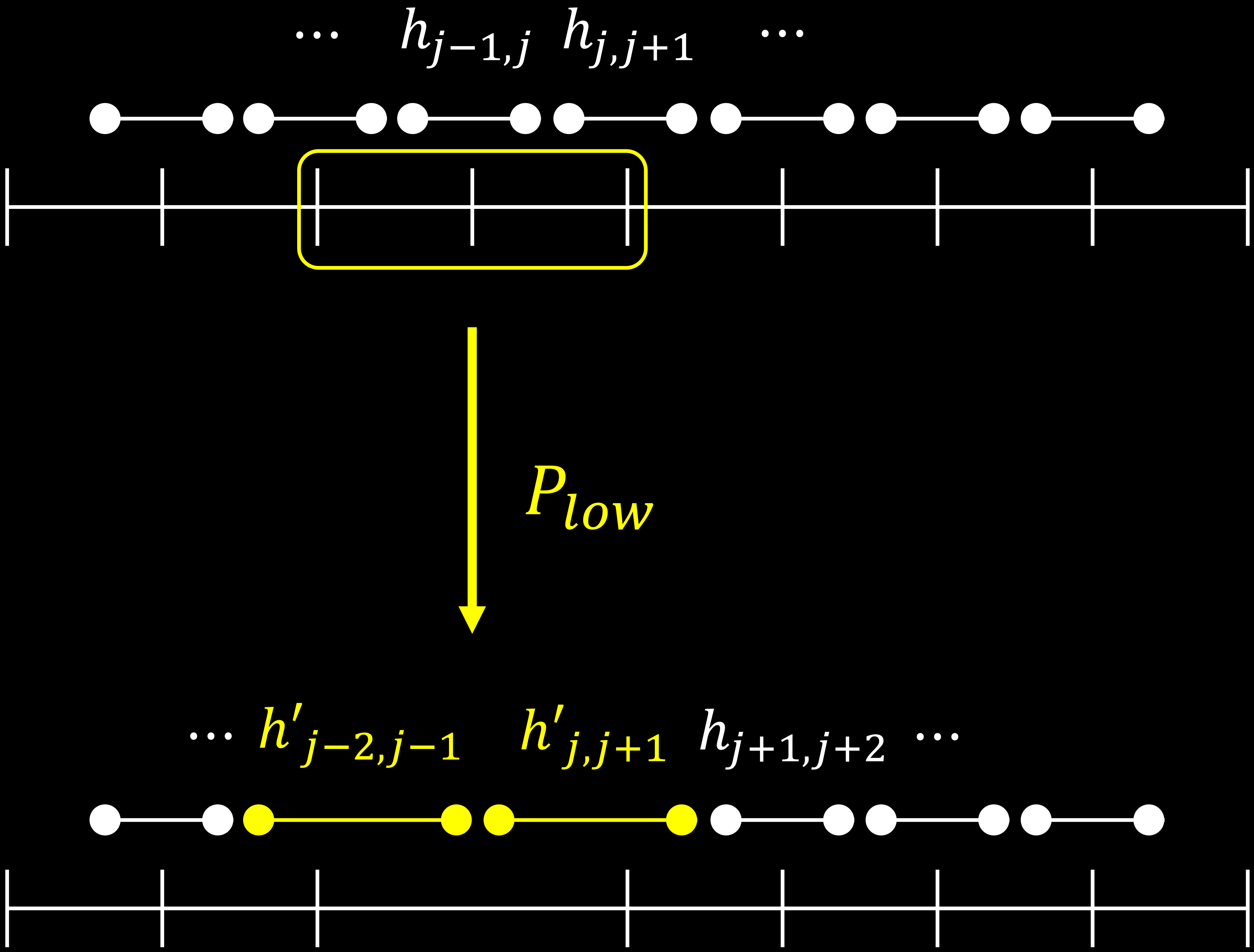
$$H(\Lambda) = \sum_j h_{j,j+1}(\Lambda)$$



$$h_{j,j+1} = \sum_{\alpha=1}^{d^2} |\psi_\alpha\rangle\langle\psi_\alpha|$$

$$\downarrow$$

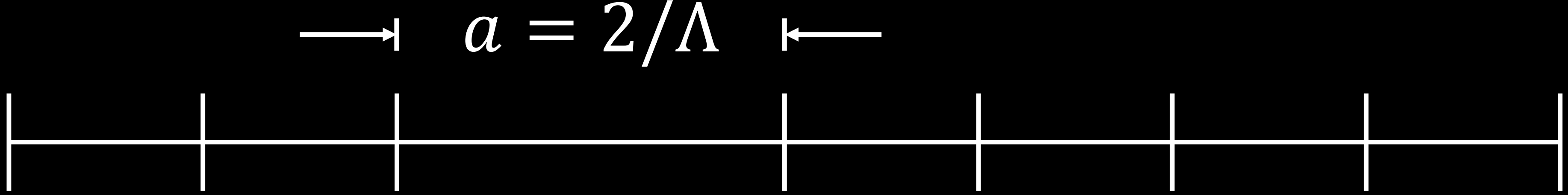
$$P_{low} = \sum_{\alpha=1}^{d} |\psi_\alpha\rangle\langle\psi_\alpha|$$

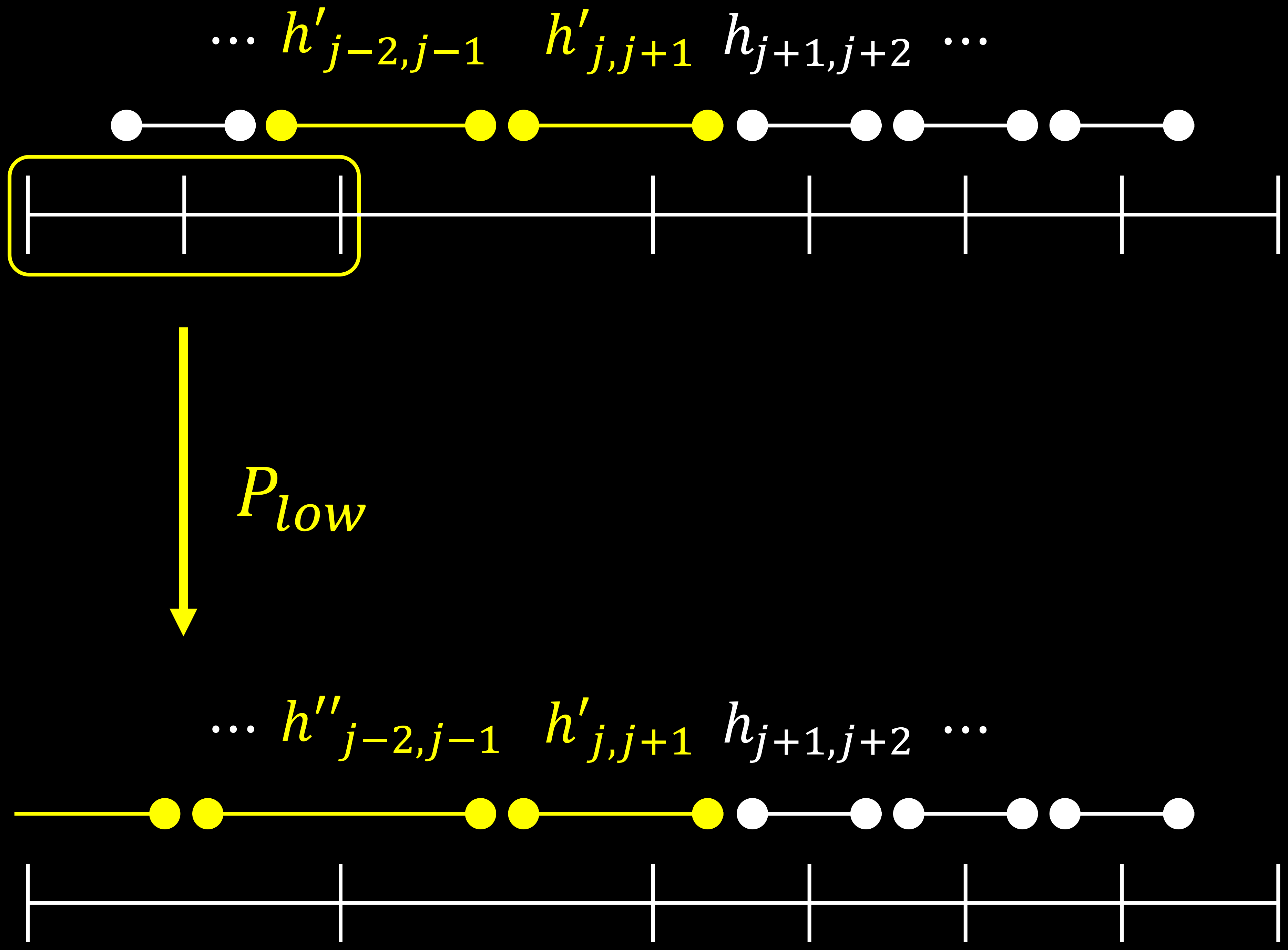


$\cdots \quad h_{j-1,j} \quad h_{j,j+1} \quad \cdots$
P_{low}
$\cdots \quad h'_{j-2,j-1} \quad h'_{j,j+1} \quad h_{j+1,j+2} \quad \cdots$

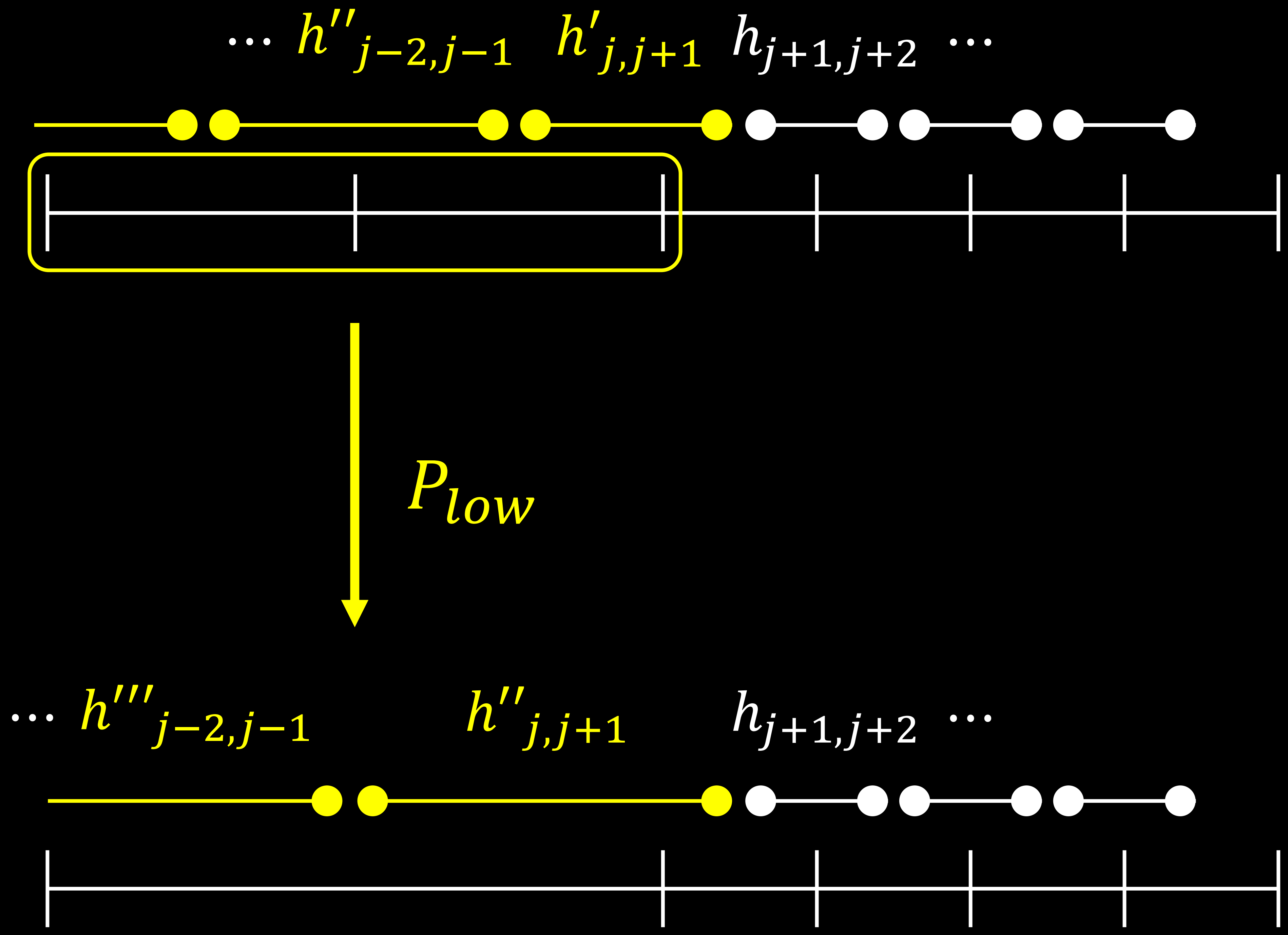
$$H_{\mathrm{eff}} = \sum_{k \neq j-2, j-1, j} h_{k,k+1} + P_{\mathrm{low}} \left(h_{j-2,j-1} + h_{j-1,j} + h_{j,j+1} \right) P_{\mathrm{low}}$$

$$\mathcal{H}_{\mathrm{eff}} = P_{\mathrm{low}} \mathcal{H} \cong \left(\mathbb{C}^d \right)^{\otimes n-1}$$

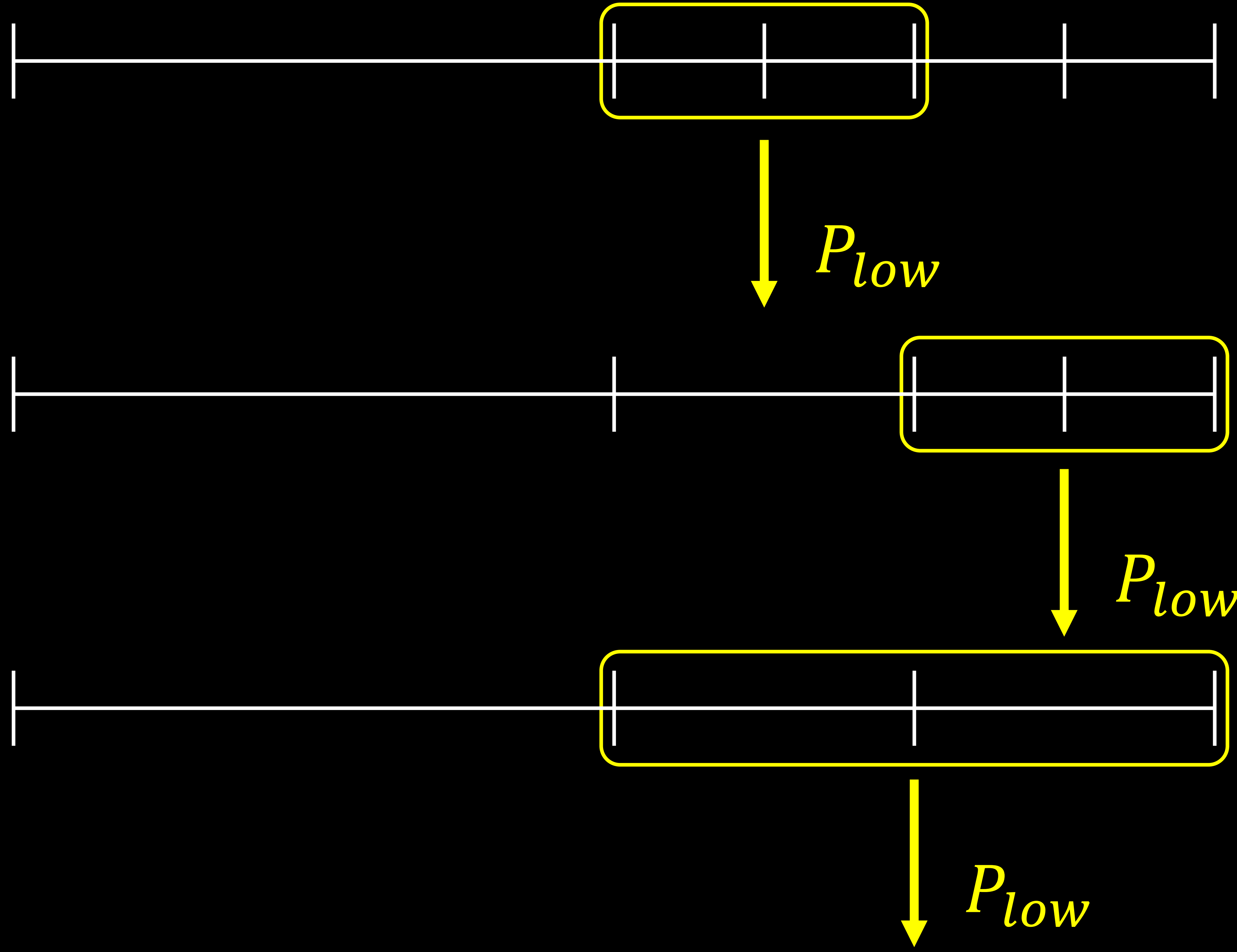




$\cdots\ h'_{j-2,j-1}\quad h'_{j,j+1}\quad h_{j+1,j+2}\ \cdots$
P_{low}
$\cdots\ h''_{j-2,j-1}\quad h'_{j,j+1}\quad h_{j+1,j+2}\ \cdots$



$\cdots\ h''_{j-2,j-1}\quad h'_{j,j+1}\quad h_{j+1,j+2}\ \cdots$
P_{low}
$\cdots\ h'''_{j-2,j-1}\quad h''_{j,j+1}\quad h_{j+1,j+2}\ \cdots$



P_{low}
P_{low}
P_{low}

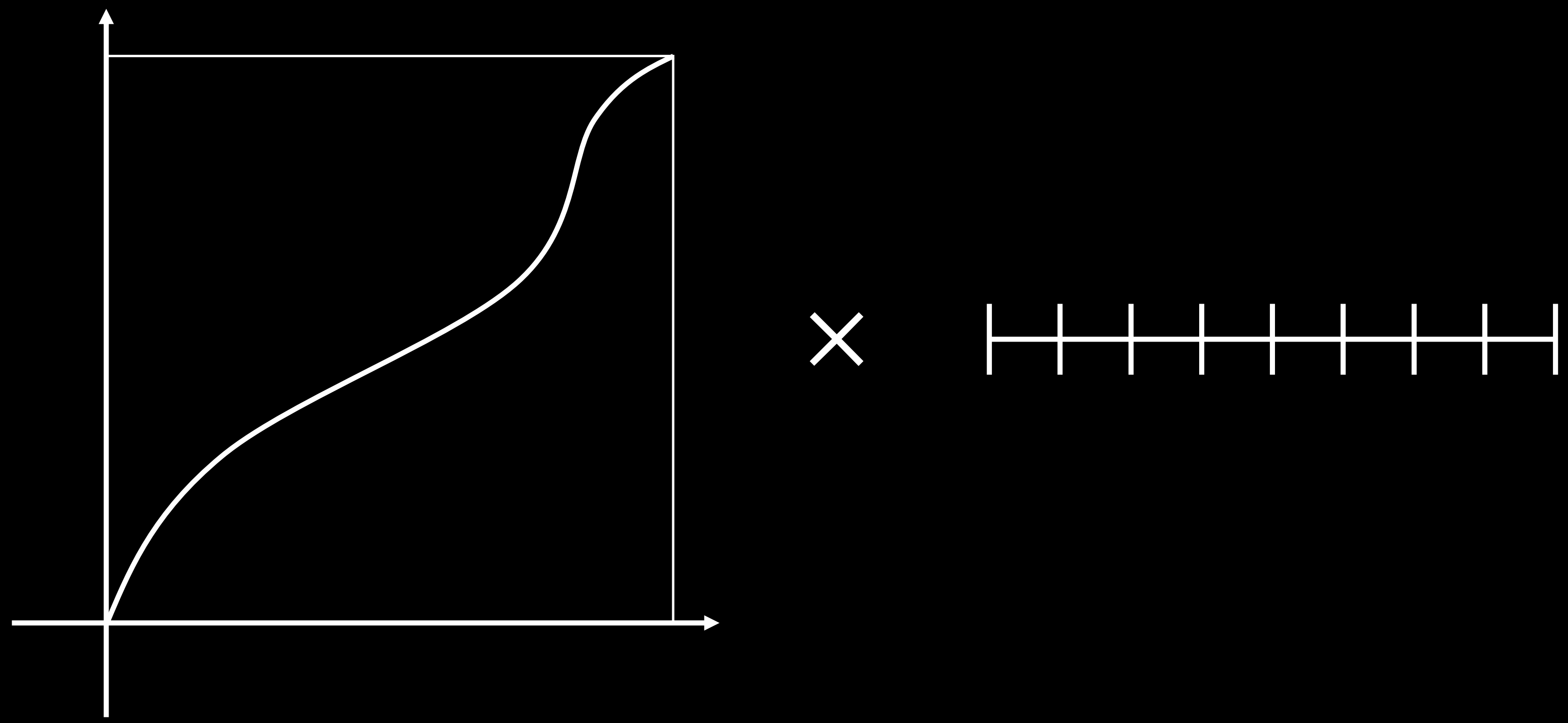
$a = 2^n/\Lambda$

$$\mathcal{H}_{\text{eff}}^{(n)} = P_{\text{low}} \cdots P_{\text{low}} \mathcal{H} \cong \mathbb{C}^d$$

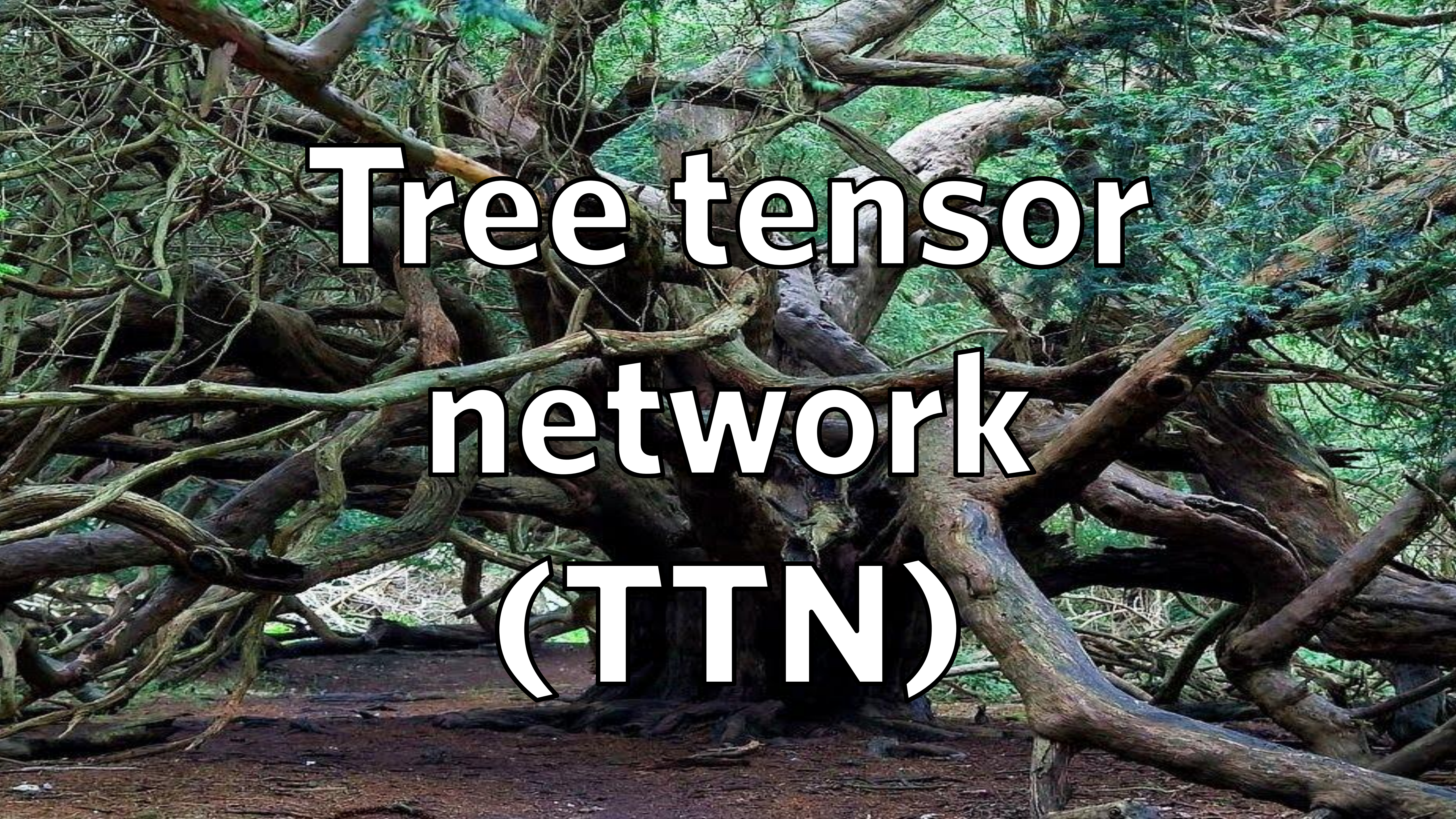
$$H_{\text{eff}}^{(n)} = h^{(n)}$$

Intermediate
lattice systems are
coarser partitions
of circle

No rescaling is applied!

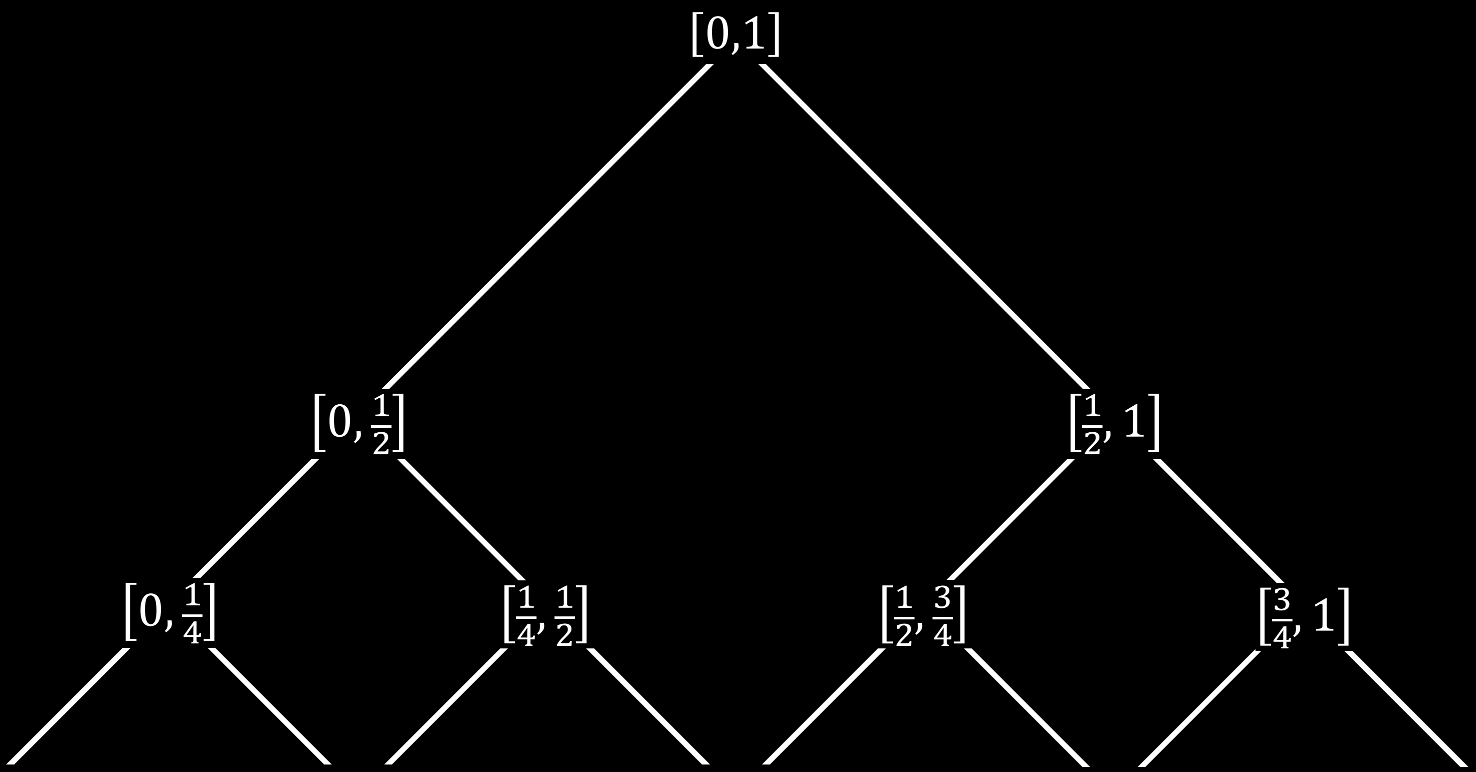


(Semi)continuous limit

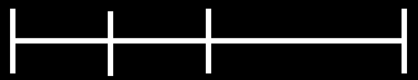
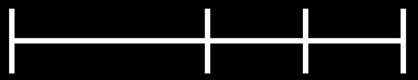

Tree tensor
network
(TTN)

Standard dyadic interval:

interval of form $\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$:



Standard dyadic partitions:
partitions $[0,1]$ into std. dyadic intervals

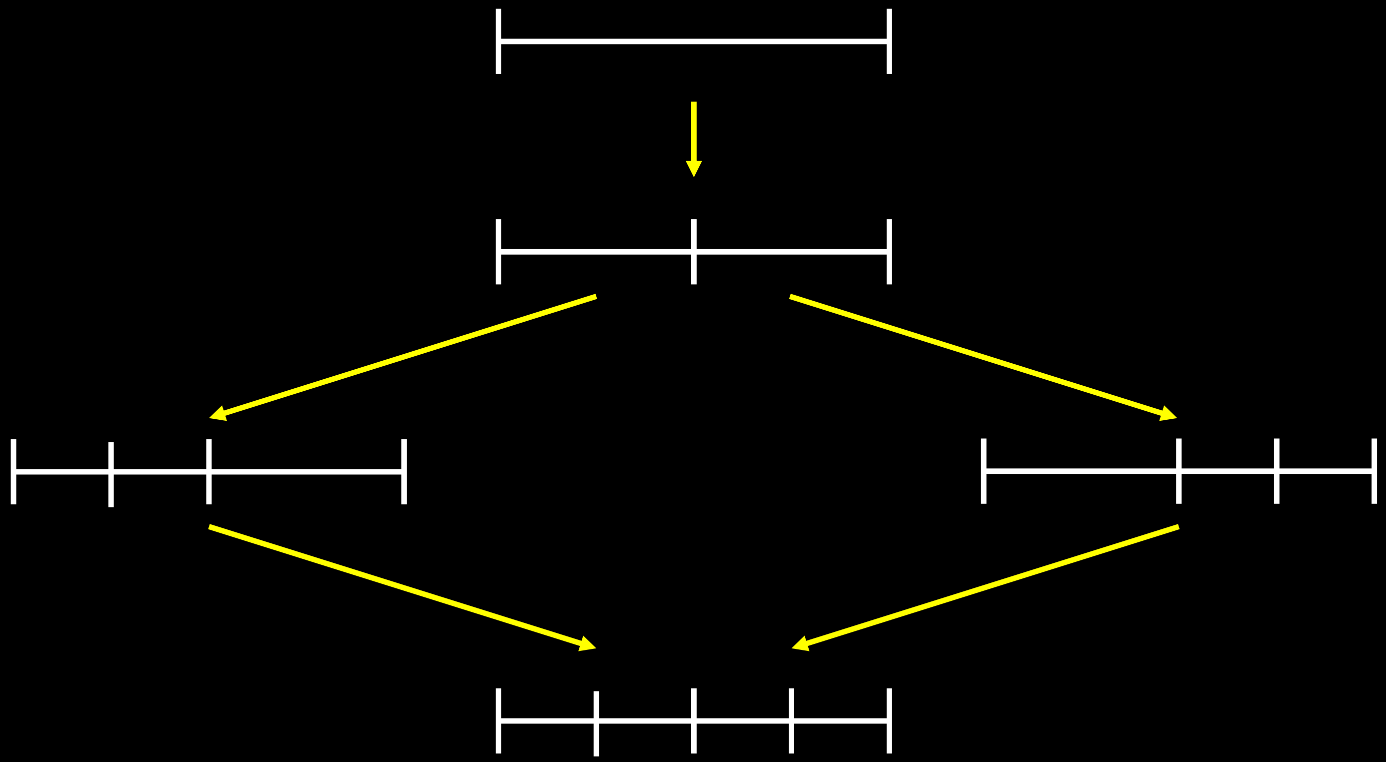
$$\mathcal{D} = \Big\{ \; , \ldots , $$  $$,$$  $$, \ldots \Big\}$$

If $P, Q \in \mathcal{D}$ say
"$P \leq Q$" to mean partition
Q is a **refinement** of P

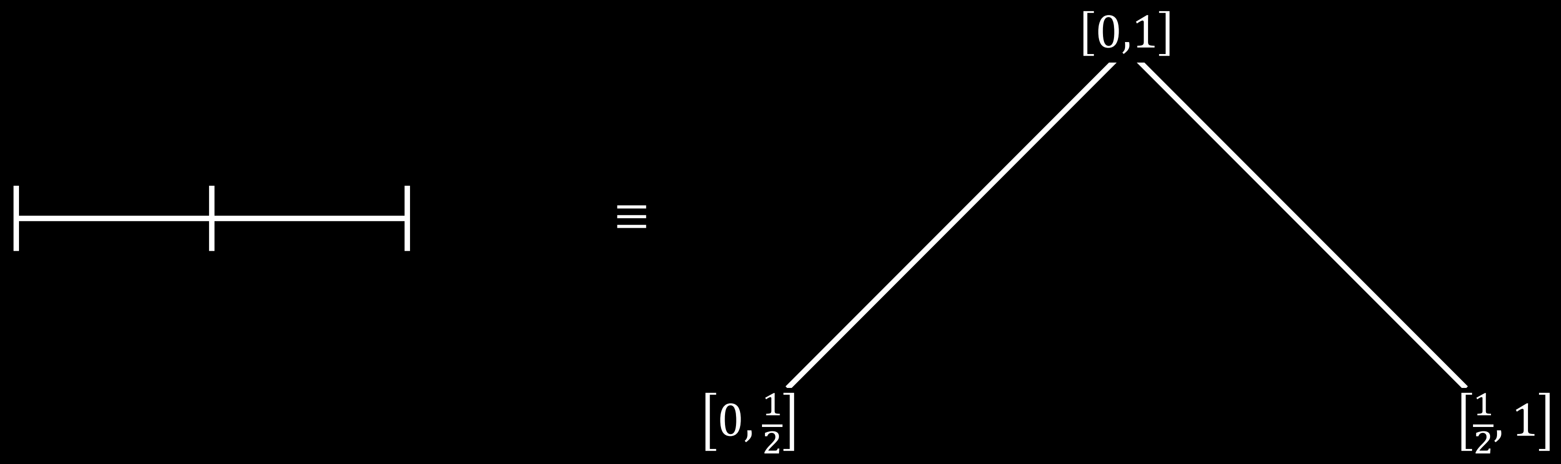
(Q has more cells)

Standard dyadic partition:

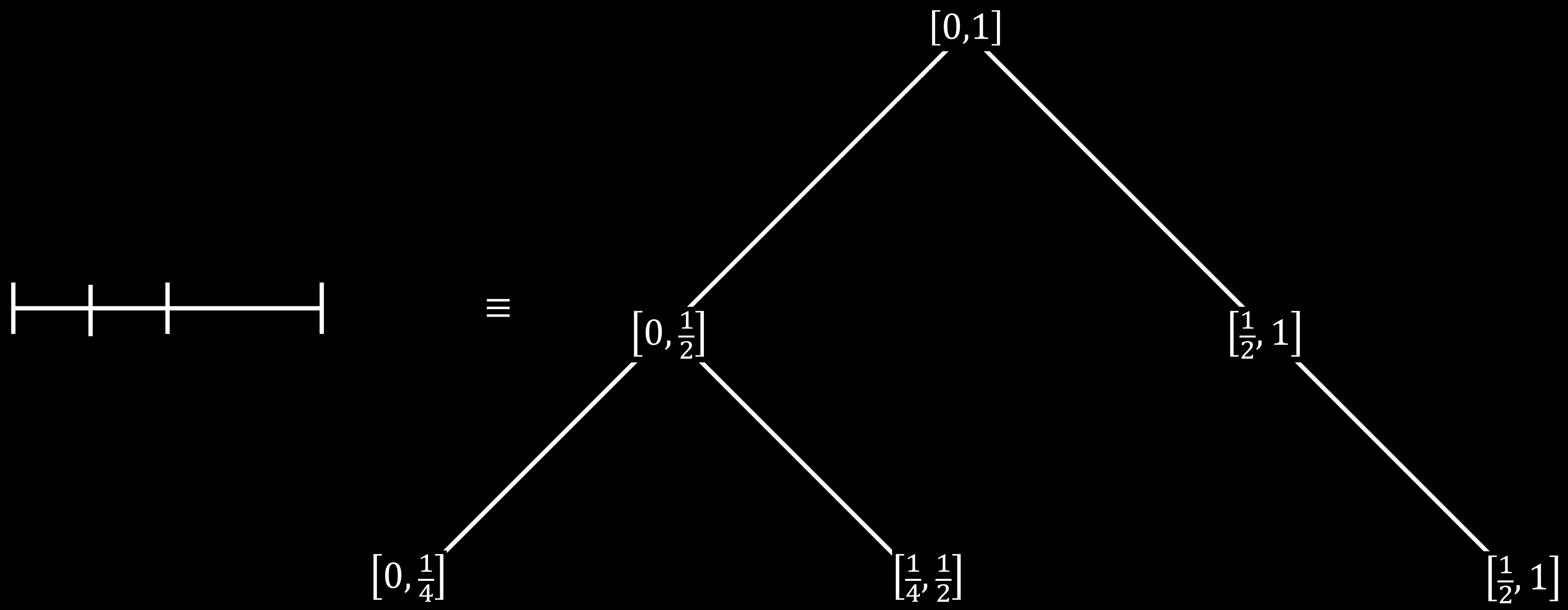
directed set $\mathcal{D}$



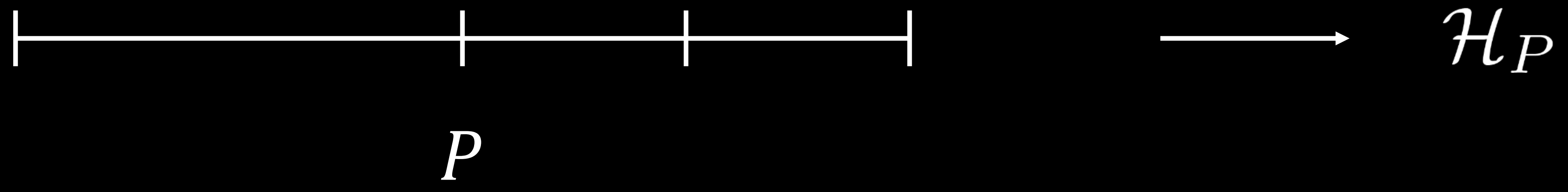
Standard dyadic partitions: representation via trees



Standard dyadic partitions: representation via trees

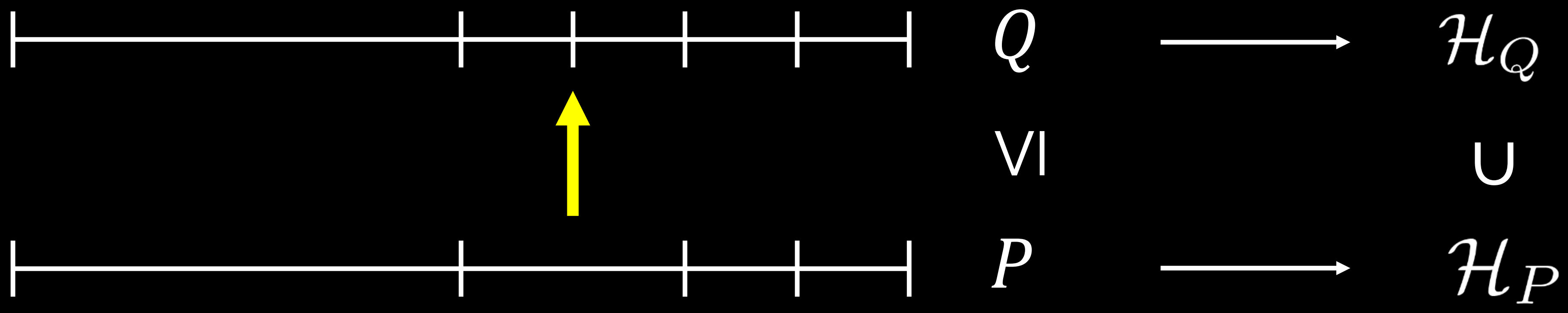


Hilbert space structure

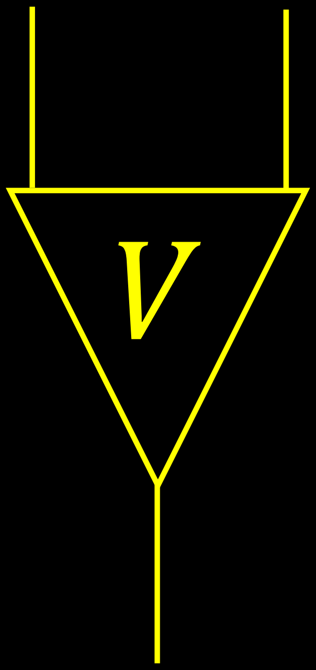


If $P \leq Q$ identify $\mathcal{H}_P \subset \mathcal{H}_Q$
via isometry:

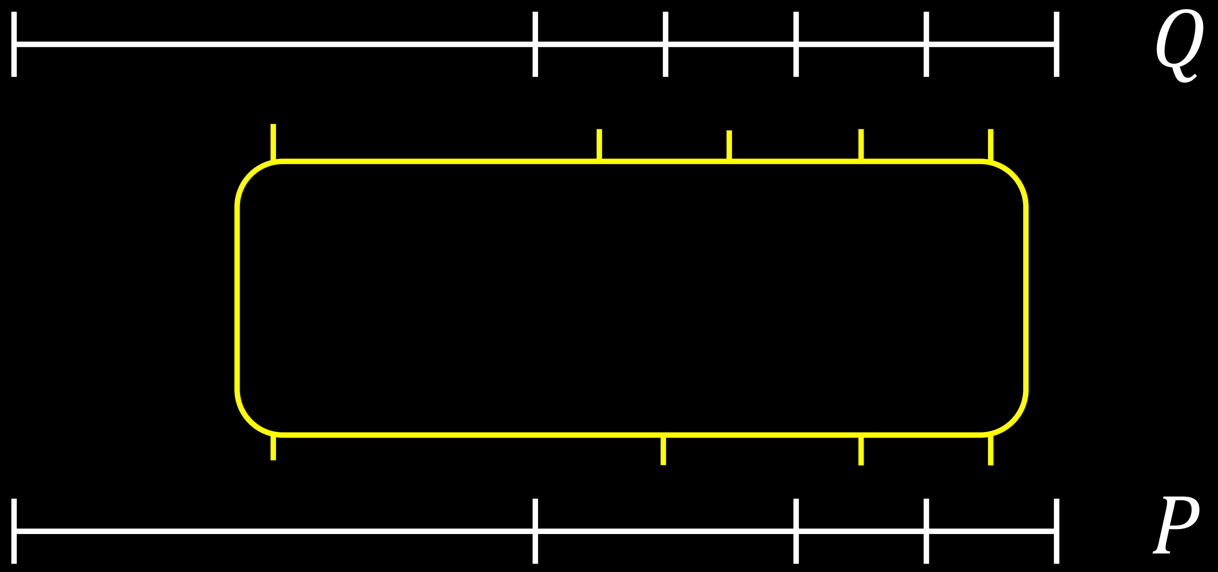
$$T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$$



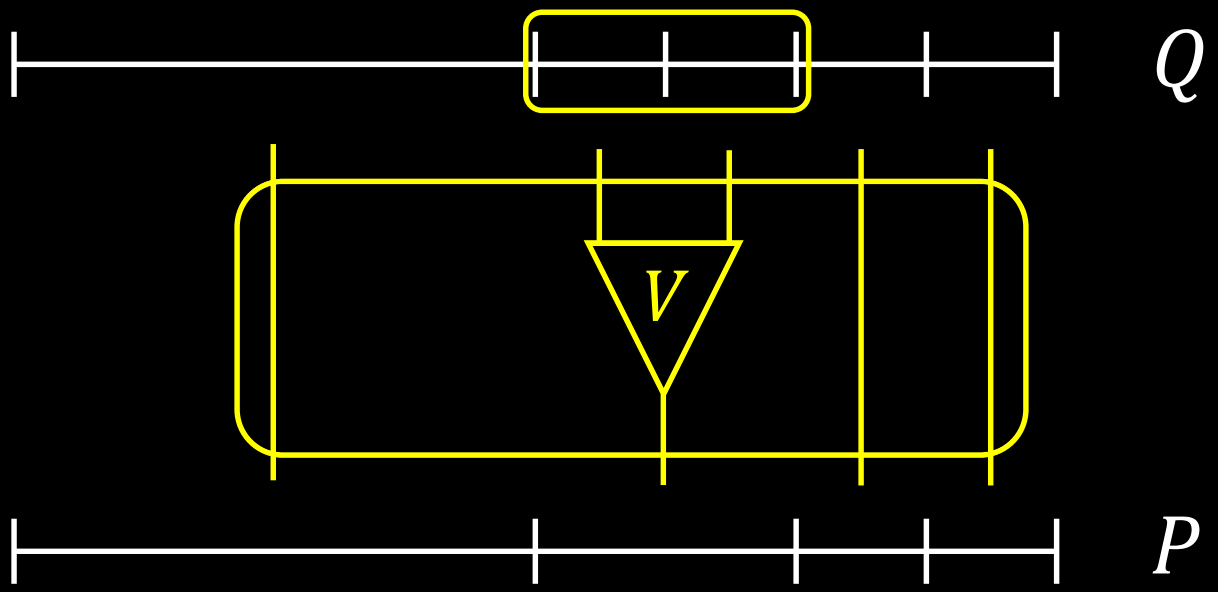
How to build isometries?

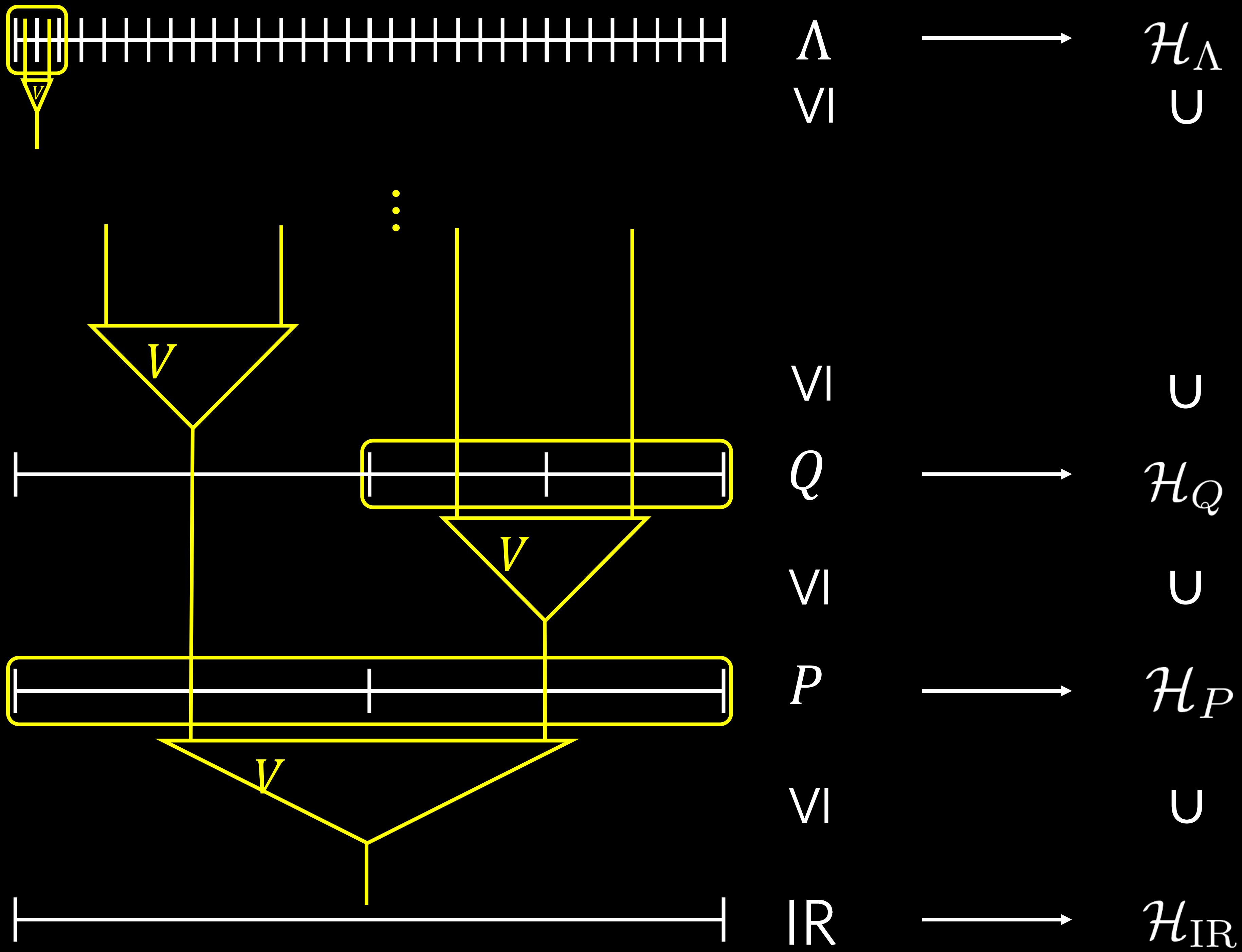


$T_Q^P =$



$$T_Q^P =$$

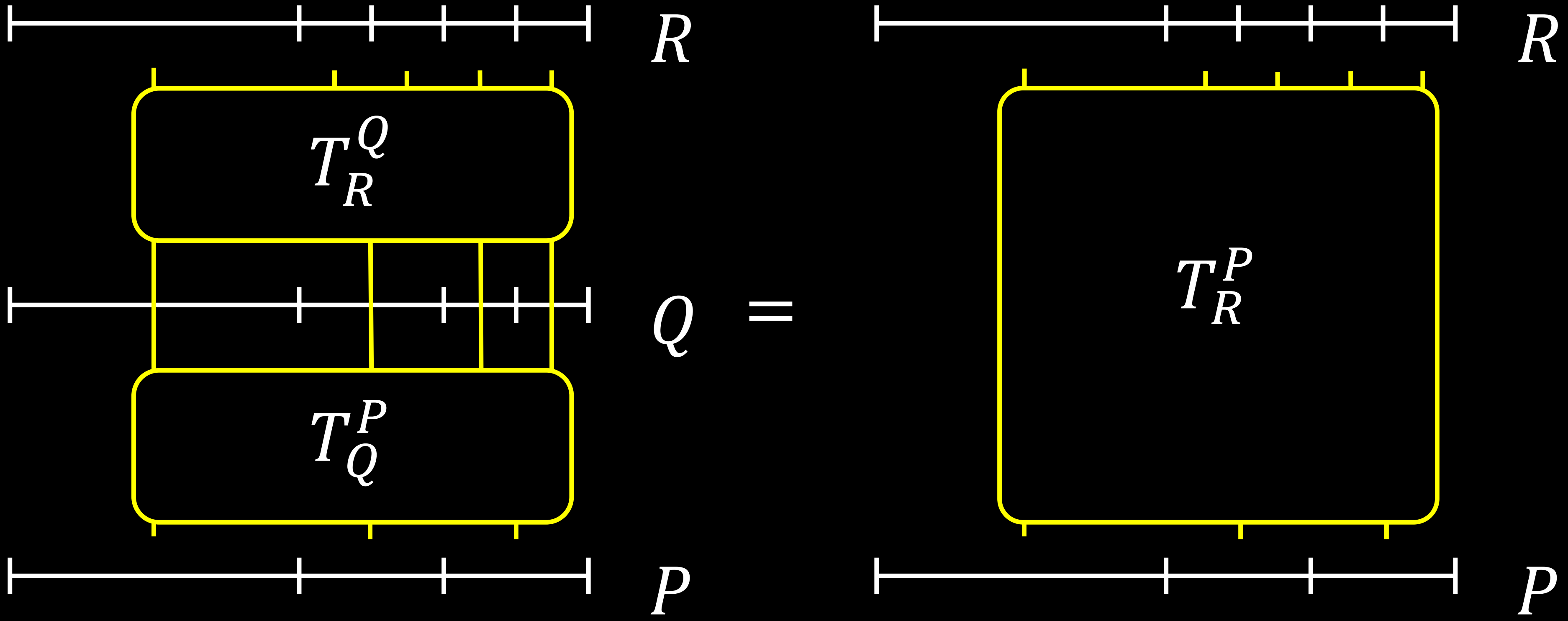




Λ
$\mathcal{H}_\Lambda$
V
VI
U
V
VI
U
Q
$\mathcal{H}_Q$
V
VI
U
P
$\mathcal{H}_P$
V
VI
U
IR
$\mathcal{H}_{IR}$

Demand WLOG

$$T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$$

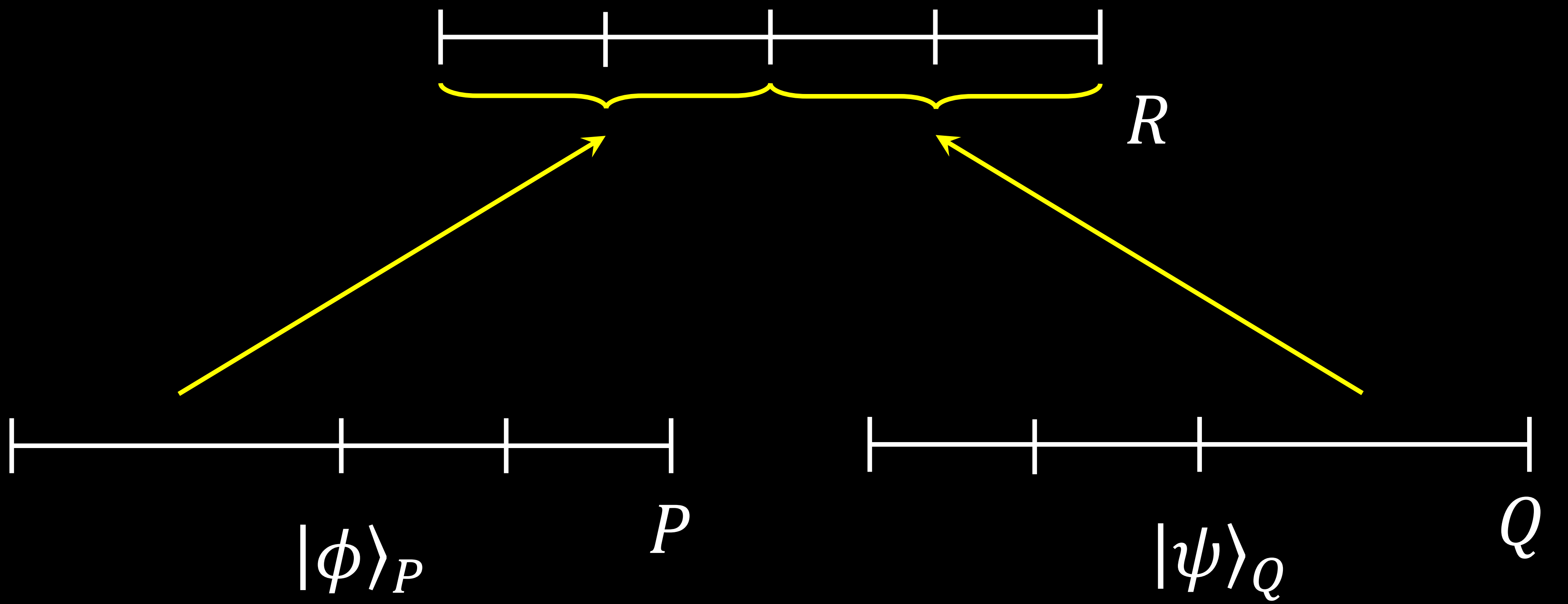


T_R^Q
T_Q^P
R
Q
P
$=$
T_R^P
R
P

Equivalence: $|\phi\rangle_P \sim |\psi\rangle_Q$

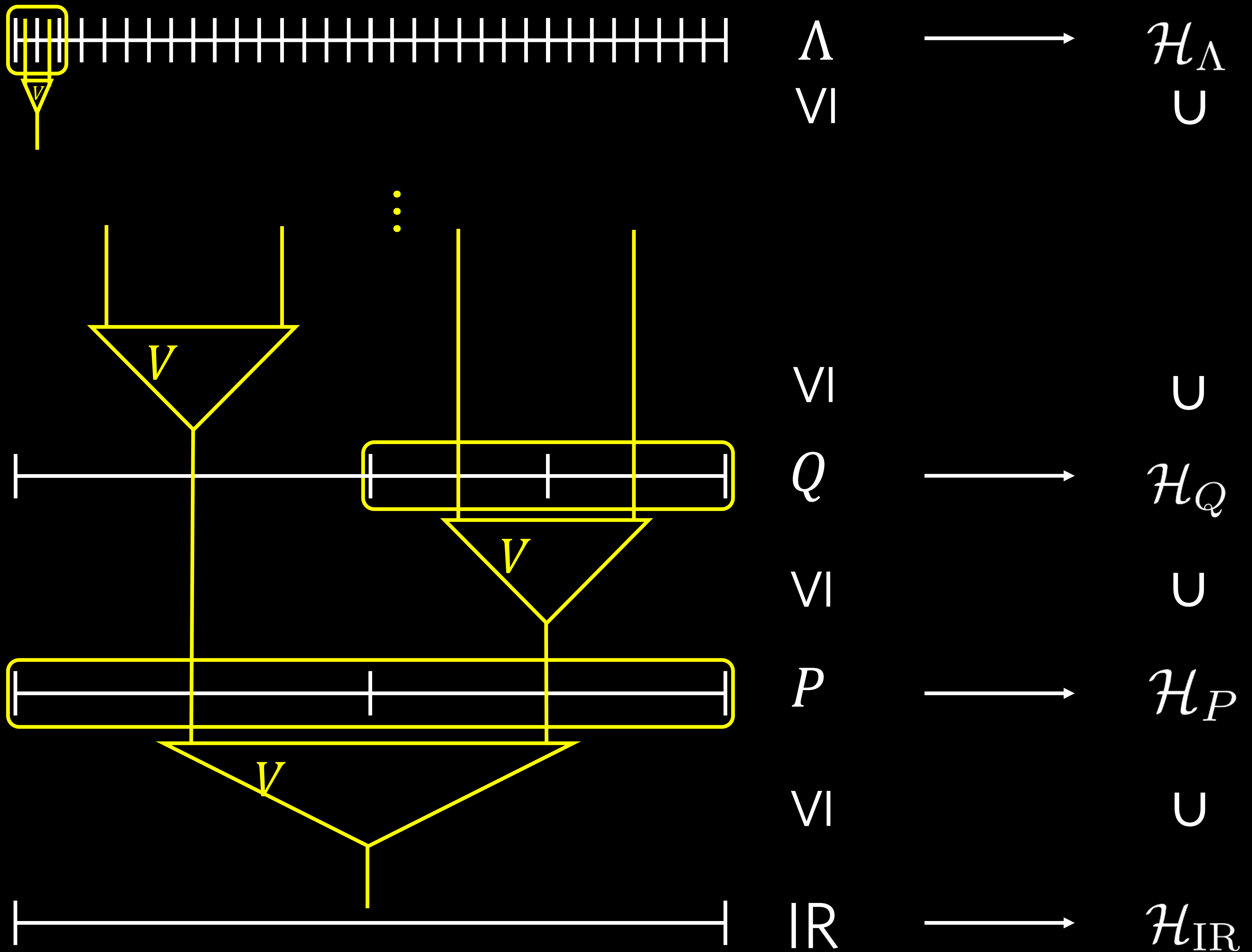
if $\exists R$

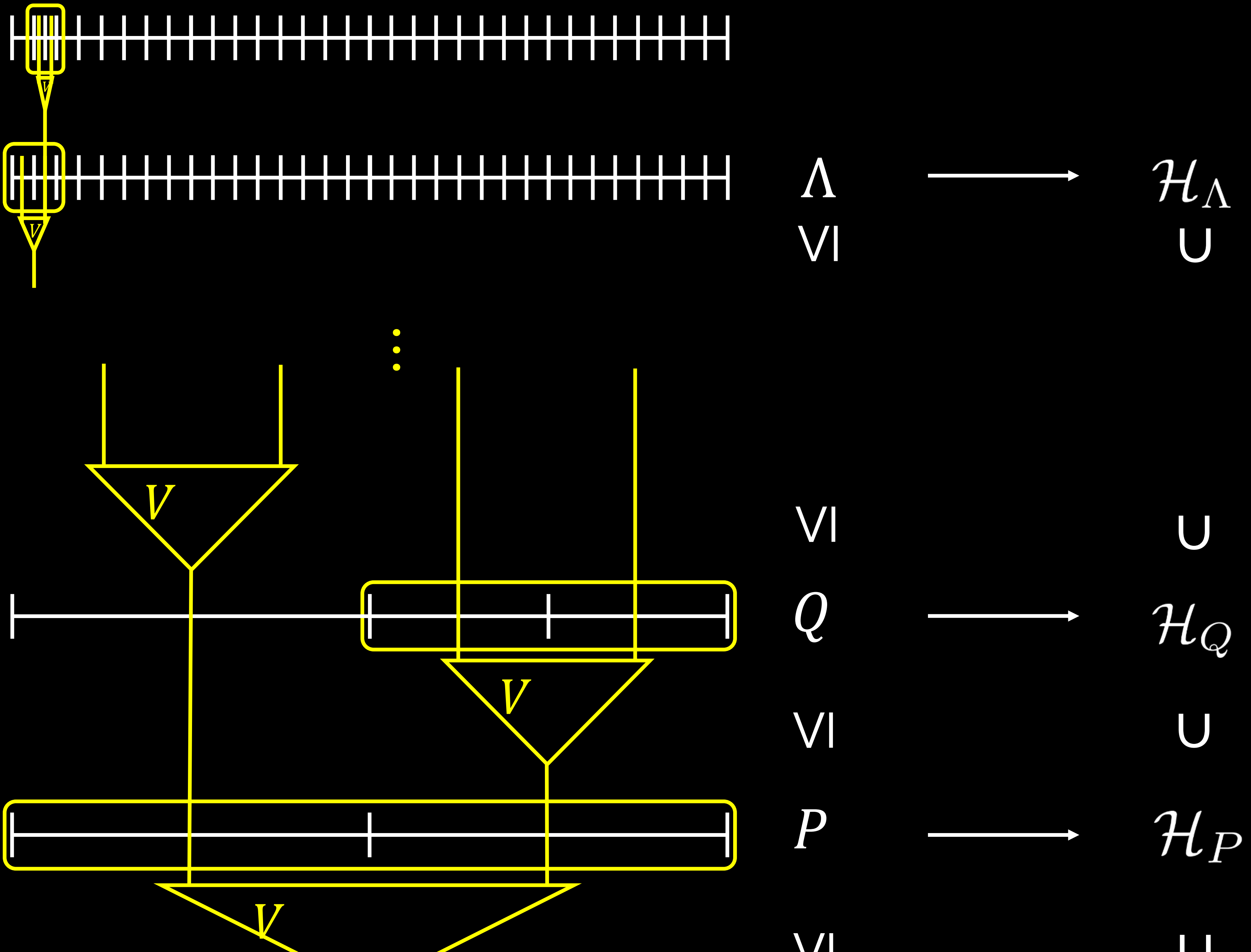
$$T_R^P |\phi\rangle_P = T_R^Q |\psi\rangle_Q$$



Semicontinuous limit: Extrapolate!

T_Q^P embeds into arbitrarily fine (std. dyadic) lattices


V
V
V
V
$\Lambda \longrightarrow \mathcal{H}_\Lambda$
$\vee\!| \quad \cup$
$\vee\!| \quad \cup$
$Q \longrightarrow \mathcal{H}_Q$
$\vee\!| \quad \cup$
$P \longrightarrow \mathcal{H}_P$
$\vee\!| \quad \cup$
$\mathrm{IR} \longrightarrow \mathcal{H}_{\mathrm{IR}}$



V
V
V
V
V
Λ
𝓗_Λ
Q
𝓗_Q
P
𝓗_P

Definition: let $(\mathcal{D}, \leq)$ be a directed set. Let a hilbert space $\mathcal{H}_P$ be given for each $P \in \mathcal{D}$ For all $P \leq Q$ let $T_Q^P : \mathcal{H}_P \to \mathcal{H}_Q$ be an isometry such that:

(1) T_P^P is the identity
(2) $T_R^Q T_Q^P = T_R^P, \quad \forall P \leq Q \leq R$

Then $(\mathcal{H}_P, T_Q^P)$ is a **directed system**.

Semicontinuous limit:

$$\widehat{\mathcal{H}} \equiv \varinjlim_{P \in \mathcal{P}} \mathcal{H}_P$$

$$= \left\{ \begin{array}{l} \text{the disjoint union of } \mathcal{H}_P \text{ over all } P \in \mathcal{P} \\ \text{modulo the equivalence relation } |\phi\rangle_P \sim |\psi\rangle_Q \\ \text{if there is } R \geq P \text{ and } R \geq Q \text{ such that} \\ \qquad T_R^P |\phi\rangle_P = T_R^Q |\psi\rangle_Q \end{array} \right\}$$

V. F. R. Jones, arXiv:1412.7740 (2014)

Residents of $\widehat{\mathcal{H}}$:

$$[|\psi\rangle_P] \equiv \{|\phi\rangle_Q = T_Q^P |\psi\rangle_P\}$$

Each hilbert space $\mathcal{H}_P$ is a natural subspace of $\widehat{\mathcal{H}}$:

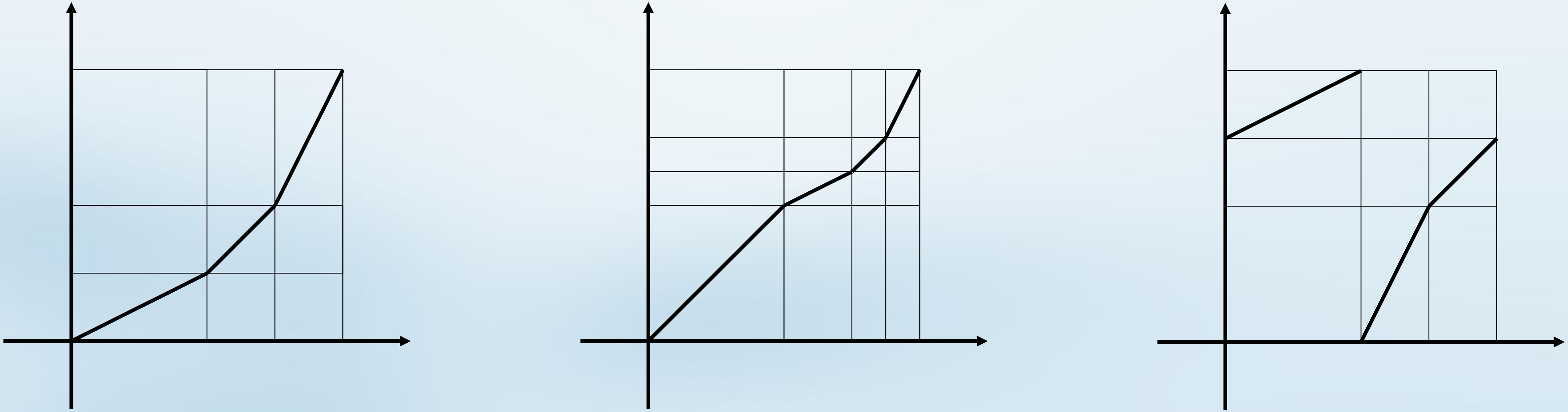
$$\mathcal{H}_P \hookrightarrow \widehat{\mathcal{H}}$$

via

$$|\psi\rangle_P \longmapsto [|\psi\rangle_P]$$

Dynamics

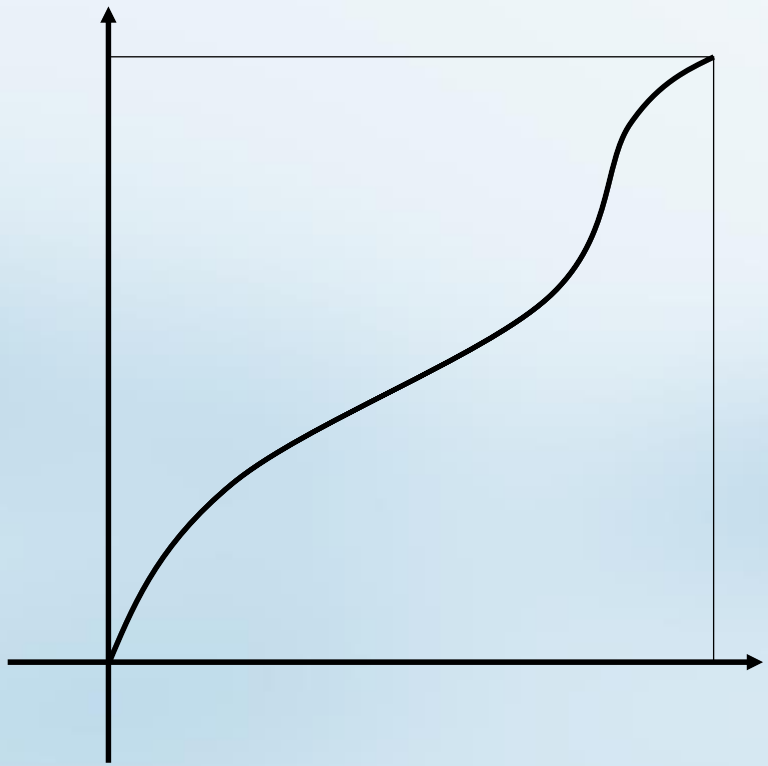
Thompson's group *T*: generated by $A(x)$, $B(x)$, and $C(x)$ under composition



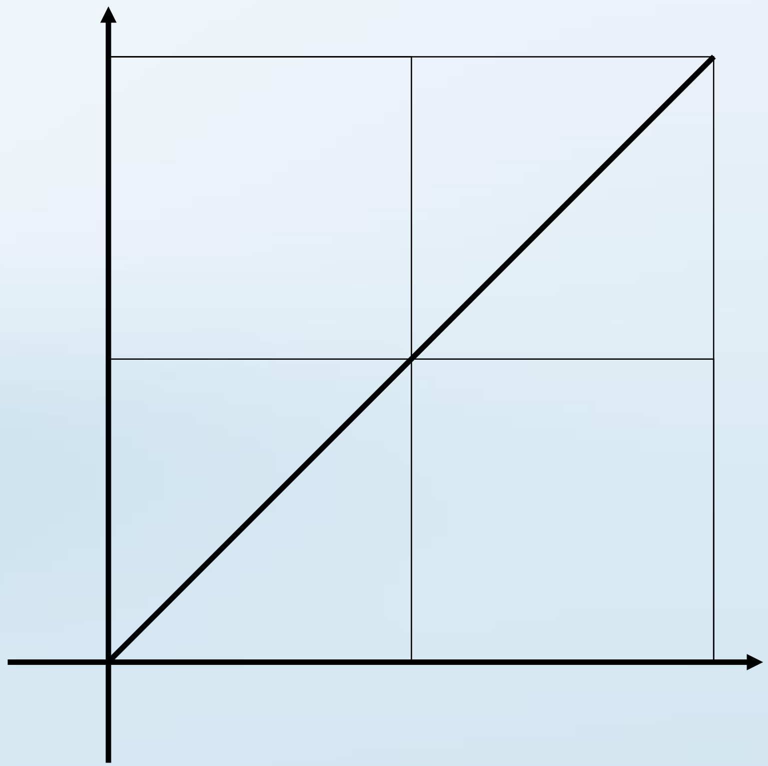
$A(x)$ $B(x)$ $C(x)$

J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to \mathbf{0}$.



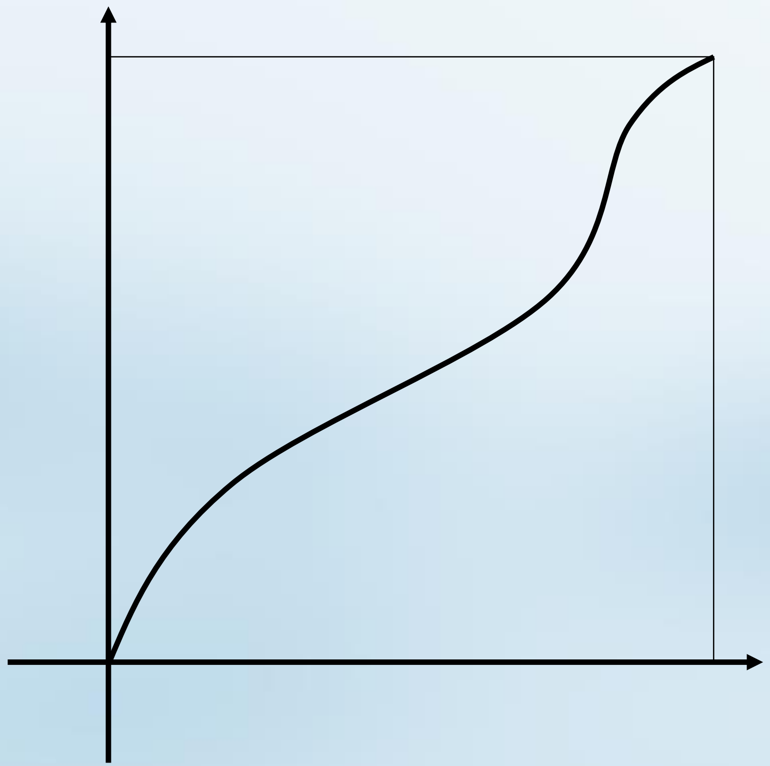
$f(x)$



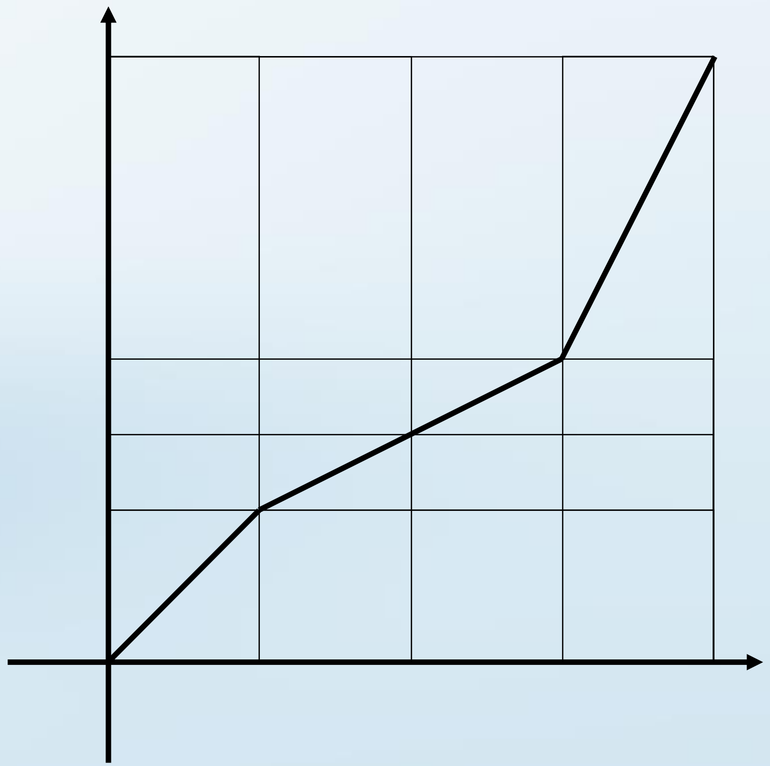
$A_1(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to \mathbf{0}$.



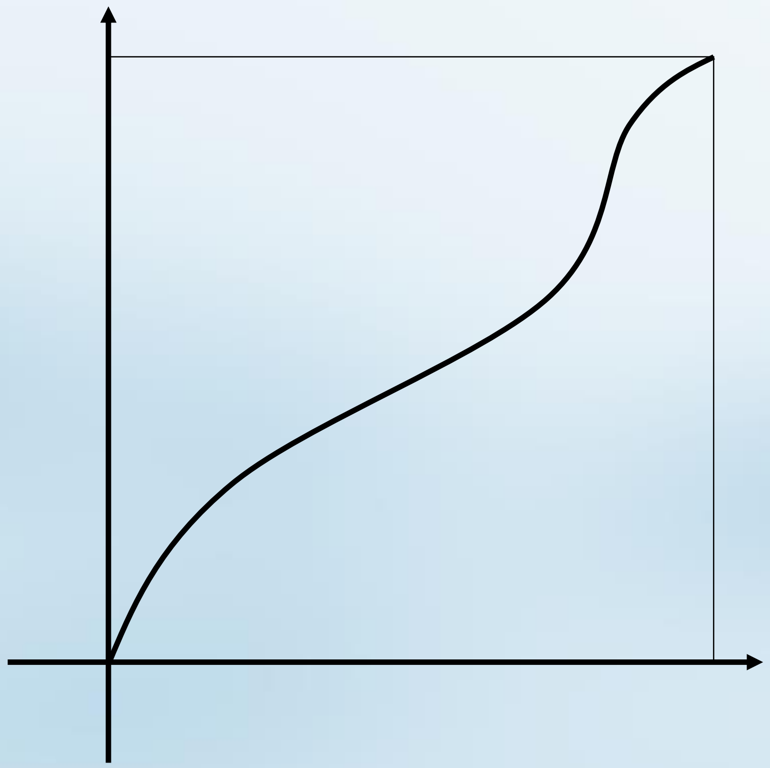
$f(x)$



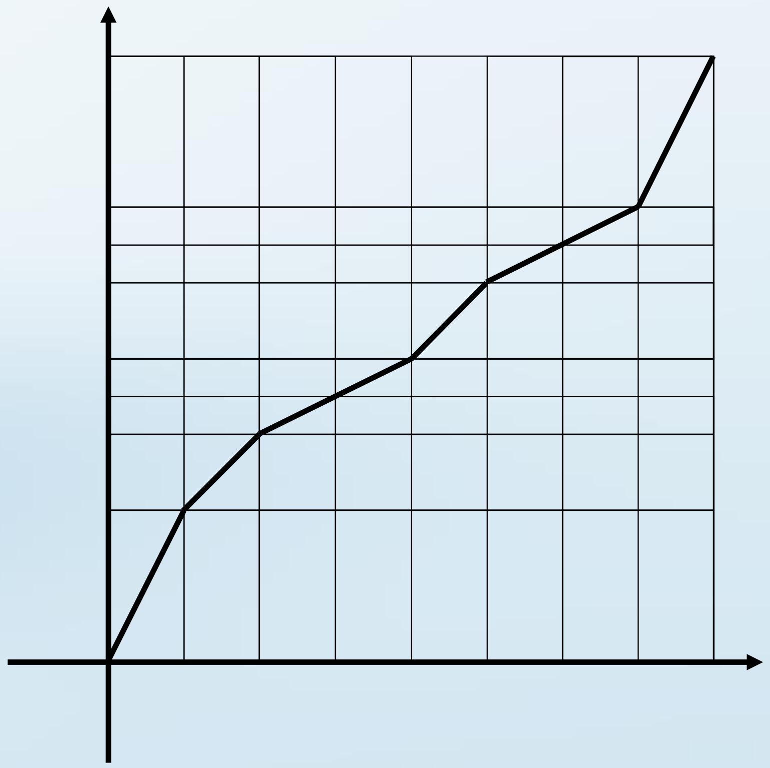
$A_2(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

Proposition ("well known"): let $f \in \mathrm{diff}_+(S^1)$. Then $\exists$ sequence $A_n(x) \in T$ s.t. $\|A_n - f\|_\infty \to 0$.



$f(x)$

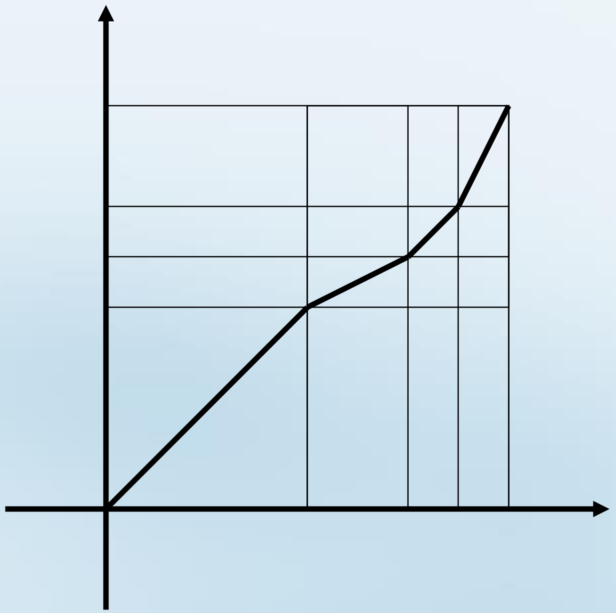


$A_3(x)$

see e.g., D. Stiegemann , arXiv:1810.11041

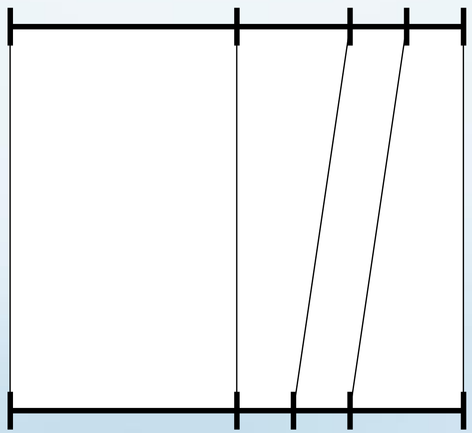
Elements of $\boldsymbol{F}$ and $\boldsymbol{T}$

Pairs of std. dyadic partitions/trees

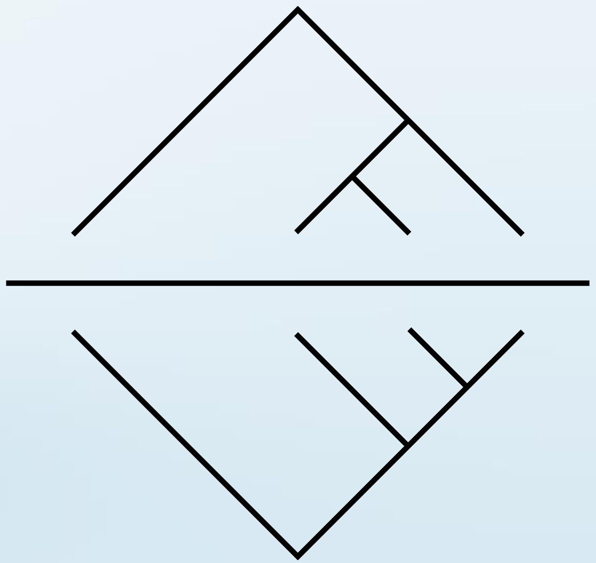


$B(x)$

$\equiv$

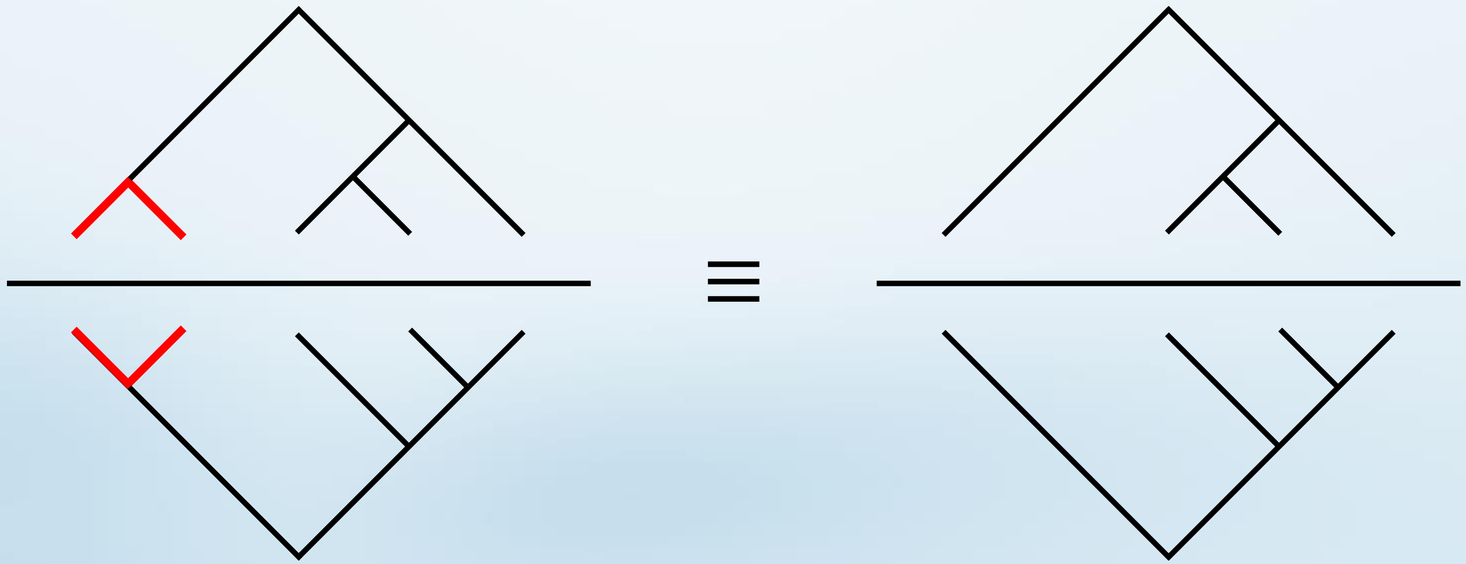


$\equiv$



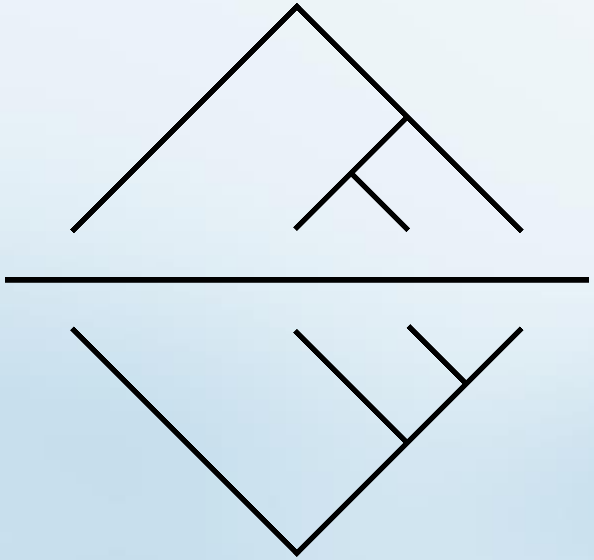
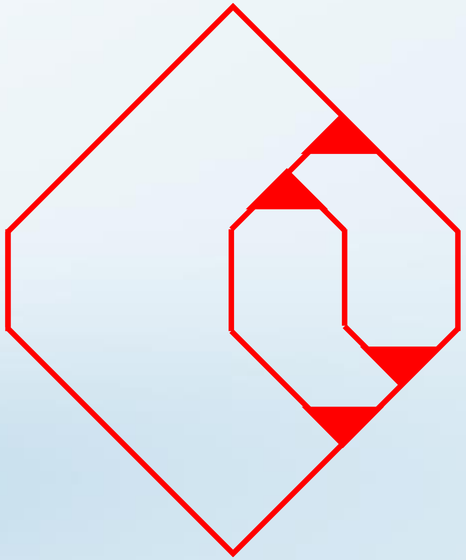
J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

Pairs of std. dyadic partitions/trees



J. W. Cannon, W. J. Floyd, and W. R. Parry, *Enseign. Math.*, vol. 42, no. 3–4, pp. 215 – 256, 1996

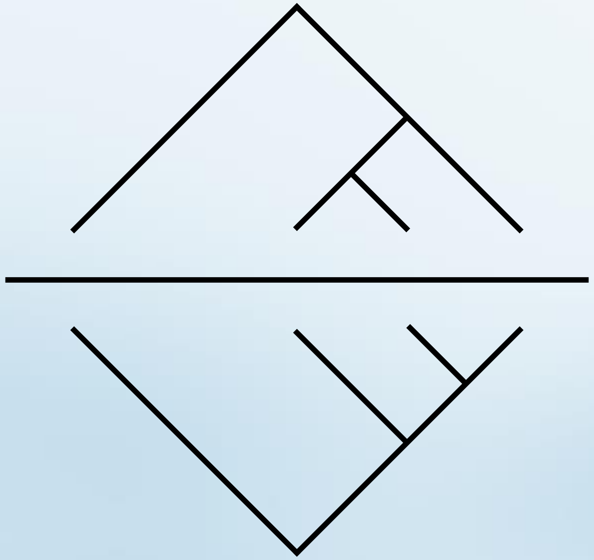
Representing F and T on $\widehat{\mathcal{H}}$

$f =$  $\rightarrow$  $\equiv \langle \Omega | U(f) | \Omega \rangle$

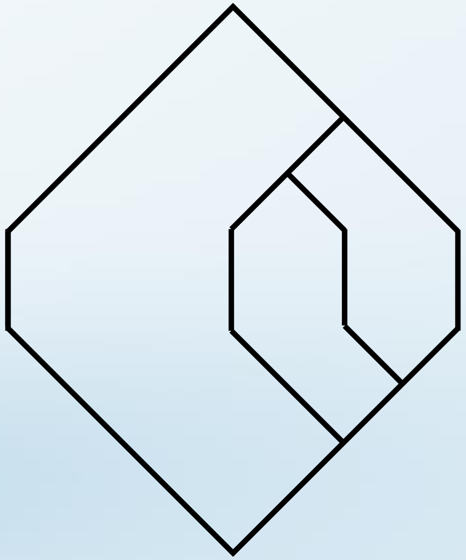
V. Jones (2014)

Representing F and T on $\widehat{\mathcal{H}}$

$f =$



$\rightarrow$



$\equiv \langle \Omega | U(f) | \Omega \rangle$

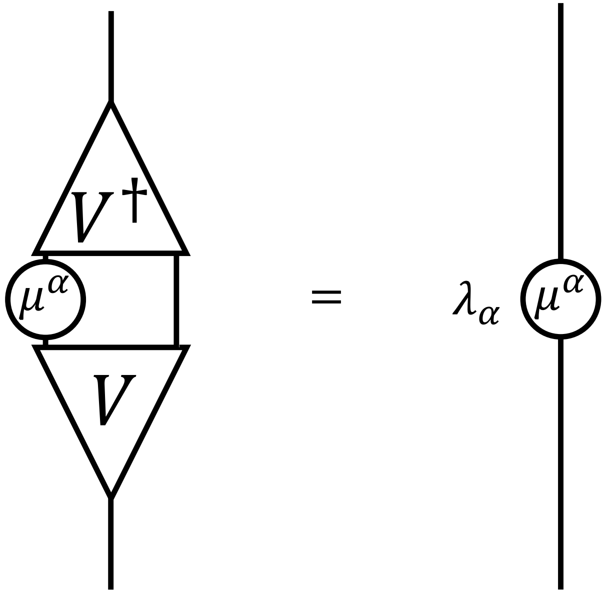
V. Jones (2014)

Observables:

“Thompson field theory”

Definition: an *ascending operator* $\mu_\alpha \in \mathcal{B}(\mathcal{H})$ is an eigenvector of the ascending channel:

$$V^\dagger(\mu^\alpha \otimes \mathbb{I})V = \lambda_\alpha \mu^\alpha$$



Definition: the *discretised field operator of type* α at $z \in S^1$ with respect to a partition $P \equiv (I_1, I_2, \dots, I_n)$ is

$$\phi_P(z) \equiv \sum_{I \in P} \mathbf{I}[z \in I](\lambda_\alpha)^{\log_2(|I|)} \mu_I^\alpha$$

TJO and D. Stiegemann, arXiv:1903.00318
See also A. Brothier and A. Stottmeister, arXiv:1901.0490

Definition (product of field operators): let $(x_1, x_2, \dots, x_n)$ be a tuple of positions and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ a tuple of types, and P a partition.

$$M_P^{\boldsymbol{\alpha}}(x_1, x_2, \dots, x_n) \equiv \phi_P^{\alpha_1}(x_1)\phi_P^{\alpha_2}(x_2) \cdots \phi_P^{\alpha_n}(x_n)$$

TJO and D. Stiegemann, arXiv:1903.00318
See also A. Brothier and A. Stottmeister, arXiv:1901.0490

Theorem: the limit

$$C^{\alpha}(x_1, x_2, \dots, x_n) \equiv \lim_{P} \langle \Omega_{P'} | M_P^{\alpha}(\boldsymbol{x}) | \Omega_{P'} \rangle$$

exists and may be calculated using $O(\log(n))$ operations.

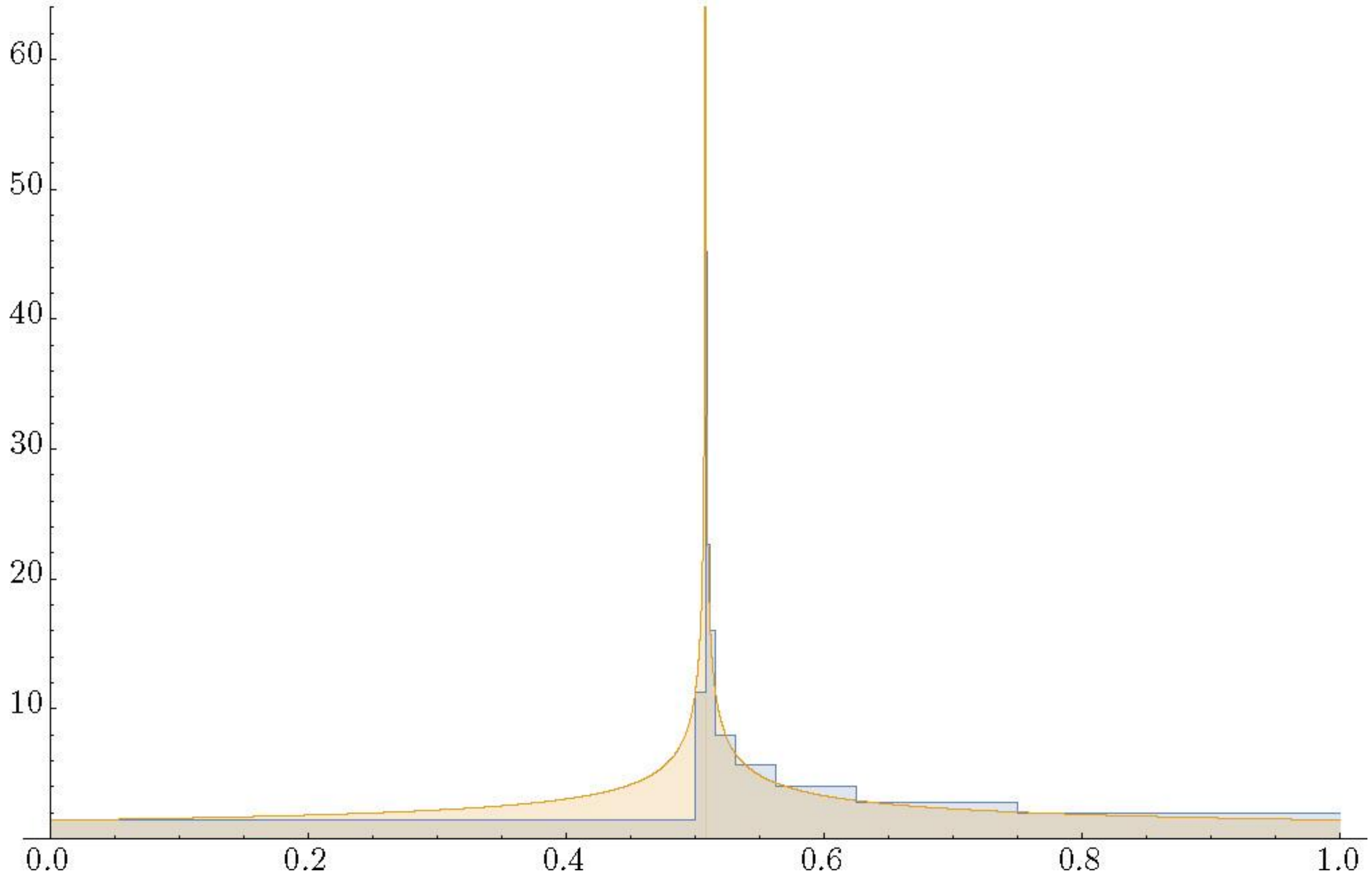
TJO and D. Stiegemann, arXiv:1903.00318

Conjecture (reconstruction):

$$C^{\alpha}(x_1, x_2, \ldots, x_n) \equiv \langle\Omega|\hat{\phi}^{\alpha_1}(x_1)\cdots\hat{\phi}^{\alpha_n}(x_n)|\Omega\rangle$$

TJO and D. Stiegemann, arXiv:1903.00318

$$C\left(\tfrac{1}{2}, x\right) \equiv \lim_{P} \langle \phi_P^{\alpha_1}\left(\tfrac{1}{2}\right) \phi_P^{\alpha_2}(x) \rangle:$$



Lemma: let x and y be two dyadic fractions

$$C^{\alpha\beta}(x,y) = \mathrm{c}(\alpha,\beta,\gamma)\, D(x,y)^{\log\lambda_\alpha + \log\lambda_\beta - \log\lambda_\gamma}$$

where $D(x,y)$ is the *coarse graining distance.*

TJO and D. Stiegemann, arXiv:1903.00318

Short distance expansion:

$$\hat{\phi}^{\alpha}(x)\hat{\phi}^{\beta}(y) \sim f_{\gamma}^{\alpha\beta} D(x,y)^{h_\gamma - h_\alpha - h_\beta} \hat{\phi}^{\gamma}(y)$$

"OPE" coefficients

TJO and D. Stiegemann, arXiv:1903.00318

Thompson field theory